Physica A 525 (2019) 33–49

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

A deterministic behaviour for realistic price dynamics

Philippe Mathieu *, Rémi Morvan

Univ. Lille, CNRS, Centrale Lille, UMR 9189 - CRISTAL - Centre de Recherche en Informatique Signal et Automatique de Lille, F-59000 Lille, France

HIGHLIGHTS

- It is possible to set prices with fully deterministic agents.
- With such agents the stylized facts of the domain are respected.
- Despite their determinism, randomness is observed at all levels of the market.
- This work illustrates that markets constitute a complex system.

ARTICLE INFO

Article history:
Received 30 August 2018
Received in revised form 10 February 2019
Available online 20 March 2019

Keywords:
Agent-based computational economics
Multi-agents systems
Artificial markets
Deterministic behaviours
Price dynamics
Microeconomics

ABSTRACT

In recent years, many studies on financial markets have relied on artificial agents, whether for the evaluation of strategies, the study of price dynamics or the efficient execution of orders. The behaviours used in those studies, often Zero-Intelligence Traders, fundamentalists or chartists, are stochastic and therefore non-deterministic, mainly because such agents easily yield a market that continuously fixes prices. We argue here that a rational and fully deterministic behaviour is sufficient both to reproduce the classic stylized facts of the field, but also to ensure that agents with different initial parameters have different opportunities to enrich themselves. To illustrate this purpose, we introduce Deterministic Artificial Traders, or DAT, and we show their performances in several situations. This result illustrates the fact that financial markets are a complex system, as some deterministic behaviours lead to some randomness in the market, both at the macroscopic and microscopic levels.

1. Introduction

Since 1987 and the first agent-based model proposed in [1], a lot of works in economics and business are based on microscopic simulations in which the artificial entities called "agents" have their own behaviour and take their decisions in complete autonomy. This current of thought is called "Agent-based Computational Economics" (ACE). The advantages of this simulation approach are undeniable, since it is the only one allowing to individualize the actors of the system, to offer a behavioural differentiation adapted to the study carried out and, above all, to allow an impact of the various previous actions on the agents themselves. It thus offers the opportunity of evaluating different behaviours both at the macroscopic level and at the microscopic level: it is then possible to obtain more explanatory models than predictive macroscopic models. Even in the simplest cases, the agent has its own cash and investments, and the various market orders have an impact on both the market and the agents themselves, thus forming the basis for the feedback loops

* Corresponding author.
E-mail addresses: philippe.mathieu@univ-lille.fr (P. Mathieu), remi.morvan@ens-paris-saclay.fr (R. Morvan).

https://doi.org/10.1016/j.physa.2019.03.042

specific to a complex system [2–4]. Several recent articles in the literature present the history of those works. One may find in [5,6] or even [7] many details about this history.

It is recalled that in a multi-agent system, agents are autonomous. They make their decisions alone, possibly perceiving their environment (market, congeners). Therefore, each agent has a single decision method to send an order. In the following work, a behaviour denotes a function that maps a set of parameters to a decision method. The general system calls this decision method fairly for each agent, either in real time (processes, threads) or in simulated time (speaking turns). Of course, the agent can, when its decision method is called, decide to not send any order. Thus lies its autonomy. For this work, we rely on the ATOM simulator[1] [8,9] who respects those different principles.

However, not all of those agent-based approaches use "real" financial orders sent to a realistic market. For example, the famous *Sante-Fe Artificial Stock Market* uses agents with a simple intention of buying or selling and therefore uses an equilibrium-based market [10,11]. The first study to use agents sending orders with prices and quantities and a double order book pricing mechanism, in line with the reality of stock markets, is the *Genoa Artificial Stock Market* [12]. As [13,14] or [15,16] did, we will also use agents sending limit orders, that are defined by an asset, a price and a direction, to a double auction order book.

Nevertheless, the literature remains rather weak on agents behaviours that are capable of sending limit orders. The overwhelming majority of the works in the literature is based on the famous Zero Intelligent Trader (or ZIT) proposed by [13,17]. Those agents have a simple but efficient behaviour and provide entropy to the market, in the sense that prices are always fixed. Another interesting point is that they are also sufficient to obtain some stylized facts well-known in the field. However, this behaviour is purely stochastic and, to the best of our knowledge, there do not exist a deterministic behaviour that, when run on a double-auction order book, yields a market that continuously fixes prices, let along respects stylized facts.

Having deterministic behaviours is desirable, both for the sake of realism and practicality, especially when one does not only want to study the market as a whole, but also to study the agents that are part of it, which can lead to interesting questions: for example, such behaviours might help us to understand why the tails of the distribution of returns are so heavy, or may even be used to know if one can discriminate between agents according to their decision method, using statistical learning tools. Those two points are more thoroughly discussed in Section 6. Moreover, by using a unique stochastic decision method for every agents, one tries to aggregate the set of every decision method followed by some trader on a (real) market. Such aggregation is unjustified and might lead to false conclusions, as explained in [18]. Though, building deterministic behaviours is far from being easy, as on a market with only deterministic and rational agents, they will tend to send the same kind of order, and no price will be set since setting a price needs both an `Ask` and a `Bid` order, on a double-auction market. Note that this problem is not present on an equilibrium-based market: if all agents send `Ask` orders or if all agents send `Bid` orders then the pricing mechanism of such a market, based on the differential between buyers and sellers, will decrease or increase the price [19], respectively.

Thus, we consider that it should be expected from an artificial market that: (i) it reproduces stylized facts, (ii) its microstructure should be identical – or at least similar – to the one of a real market and (iii) assumptions made about traders' behaviour are realistic. Hence, we propose a fully deterministic behaviour of a Deterministic Artificial Trader, or `DAT`, that is gradually refined to answer the following questions:

1. Can a deterministic behaviour lead to a market that continuously fixes prices?
2. If so, how complex should this behaviour be in order to reproduce stylized facts?
3. Does luck exist on such a market?

In other words, can randomness emerge at the macroscopic level from microscopic determinism on a double-auction order book, and can it, through the feedback loops that define complex systems, lead to some randomness at the microscopic level?

Section 2 describes the stylized facts that we study in order to see if our agents are realistic. Then 3 describes the behaviour of `DAT-F`, our deterministic and fundamentalist agent, that allows us to get a market on which prices are continuously fixed, and that respects some stylized facts. Then this behaviour is refined in Section 4 by giving some information to the agent, which defines our informed fundamentalist `DAT-IF`. This agent allows us to respect the stylized facts. In order to allow the randomness that emerged at the market level to lead to some randomness at the agent level, in Section 5, we allow our agents to take part of their decision using trends. The obtained agent is called `DAT-IFC` as it is part informed fundamentalist and part chartist. Finally, in Section 6, we discuss about the choices we made for the simulations, and then we highlight interesting perspectives.

2. Stylized facts

As stylized facts are properties common across markets, any artificial market should reproduce them in order to be a good model.

[1] Downloadable at https://github.com/cristal-smac/atom.

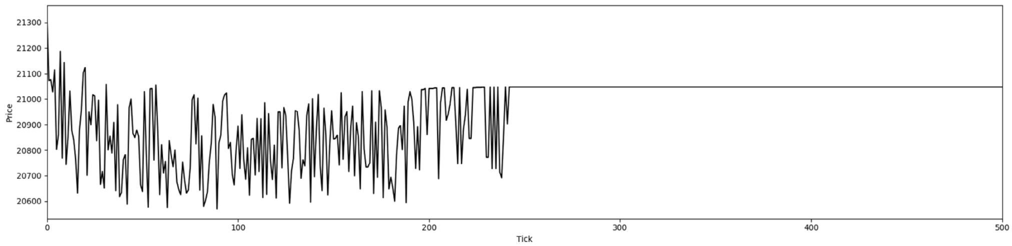


Fig. 1. Sequence of prices at the end of each tick, obtained on a market using 50 ZITs that can neither have negative cash, nor negative quantity of assets.

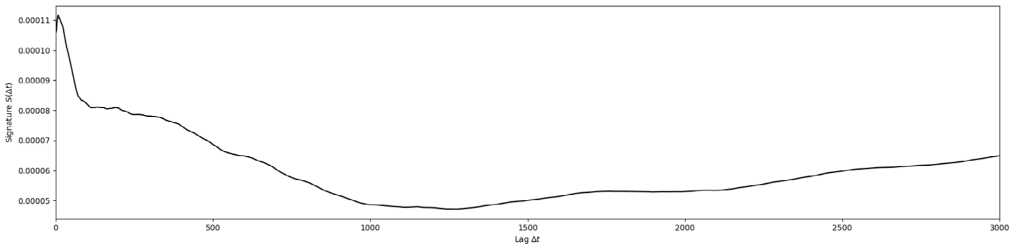


Fig. 2. Signature plot of a real price series.

Continuously fixing prices The first and rather obvious stylized fact is that a market never freezes, or, in other words, prices are continuously fixed: there is no point in time such that there will be no transaction between traders later on. If this property is easily reproduced using agents as simple as ZITs, it is interesting to notice that when strong constraints are applies to agents, satisfying it becomes less easy. For example, if one runs a double-auction market whose agents are ZITs that choose uniformly the direction, price, and quantity of the orders they send – for the price and the quantity, two fixed bounds are defined at the beginning of the simulation –, and if those ZITs can never have negative cash nor negative assets,[2] then the price series will look like what is pictured in Fig. 1: since agents cannot cancel their orders, after a certain number of steps, the system freezes.[3] When a small constraint is applied to ZITs, they no longer continuously fix prices. Henceforth, one can only imagine how hard it is to build agents that yield a market on which prices are continuously fixed when the constraint is to have deterministic agents. Note that, those agents have to be heterogeneous in order to fix a price [20].

Random walk Given a price series $(P_t)_{0\leq t\leq T}$, where each P_t is a random variable, one can define its signature plot as

$$S(\Delta t) = \frac{\mathbb{E}[(P_{t+\Delta t} - P_t)^2]}{\Delta t}$$

If $(P_t)_{0\leq t\leq T}$ follows a random walk – for example if P_{t+1} is equal to $P_t + 1$ with probability one half, and $P_t - 1$ otherwise – , then $S(\Delta t)$ will not depend on Δt – in this case, it will always be equal to 1. This means that the mean quadratic difference between two prices is proportional to the time length between those two prices. The signature plot obtained using price series of Microsoft shares is shown on Fig. 2 – for each Δt, the empirical mean was computed on 5000 prices.

Properties of logarithmic returns Most stylized facts focuses on logarithmic returns: a rather exhaustive list can be found in [21]. Given a price series $(p_t)_t$, logarithmic returns are defined as $(\log(p_{t+1}) - \log(p_t))_t$. As many articles in the literature, we decided to focus on the following properties:

- *Heavy tails* The distribution of returns have fat tails compared to a normal distribution. In particular, its kurtosis is higher than 3.
- *Aggregational Gaussianity* The distribution of $(\log(p_{t+\Delta t}) - \log(p_t))_t$ converges in probability towards a normal distribution as Δt tends to $+\infty$.
- *Absence of autocorrelations* The autocorrelation of returns is insignificant for large values of Δt
- *Slow decay of autocorrelation in absolute returns* The autocorrelation of absolute returns $((|\log(p_{t+1}) - \log(p_t)|)_t)$ decays as a power law with an exponent between -0.3 and -0.2. Fig. 3 shows the plot of the autocorrelation function of the absolute returns on a log–log scale, using the data of Microsoft shares. Note that the exponent found for this data is -0.20.

2 I.e. if the order randomly chosen by the agent could lead him to have negative cash or negative asset, it will not send the order.

3 Let us note that "tick" refers to allowing each agent to talk once, in a fair random order.

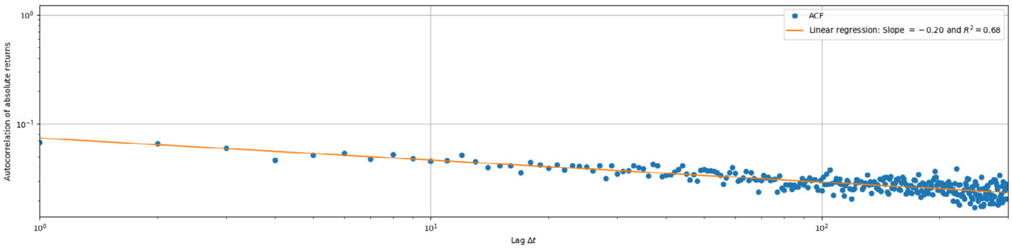


Fig. 3. Autocorrelation of absolute returns on log–log scale, for a real market.

3. A minimalist fundamentalist and deterministic agent: DAT-F

Notations: A limit order o is a triplet $(o_d,\ o_p,\ o_q)$ whose elements designate the direction of the order (Ask or Bid), its price and its quantity, respectively. When it is not necessary to specify one of the three parameters, it will be replaced by the _ character. If the agent does not wish to send an order, the convention $o = \emptyset$ is used. In addition, we will note $[\![a,\ b]\!]$ the set of integers between $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$.

This section focuses on constructing a minimalist agent, i.e. the simplest possible behaviour, that is deterministic and that makes it possible to reproduce classic stylized facts. This agent, DAT-F, is fundamentalist: it has an estimation of the fundamental value of the asset, and relies on this estimation to send its orders. Its behaviour is as follows: if the bestAsk has a lower price than the agent's estimation, then the agent will send an order Bid at the price of bestAsk. Likewise, if the bestBid has a higher price than the estimation of the agent's estimation, then the agent will send an order Ask, whose price will be equal to the price of the bestBid. In other words, the agent only agrees to buy at a price below its estimation of the fundamental value, and only to sell at a higher price. If not, the agent will try to reduce the bid-ask spread, while centring it around its estimation of the fundamental value.

3.1. Formal behaviour

For the sake of simplicity, the market that is taken here consists of a single asset, and therefore of only one order book. However, the behaviour of DAT-F can be immediately generalized to a market with several assets/order books.

An agent i has several intrinsic parameters, which will determine which order it will send, when the market allows it to talk:

- $\alpha_i \in [0, 1]$: its aggressiveness
- $\kappa_i \in \mathbb{N}^*$: its self-confidence
- $\pi_i \in \mathbb{N}^*$: its estimation of the fundamental value

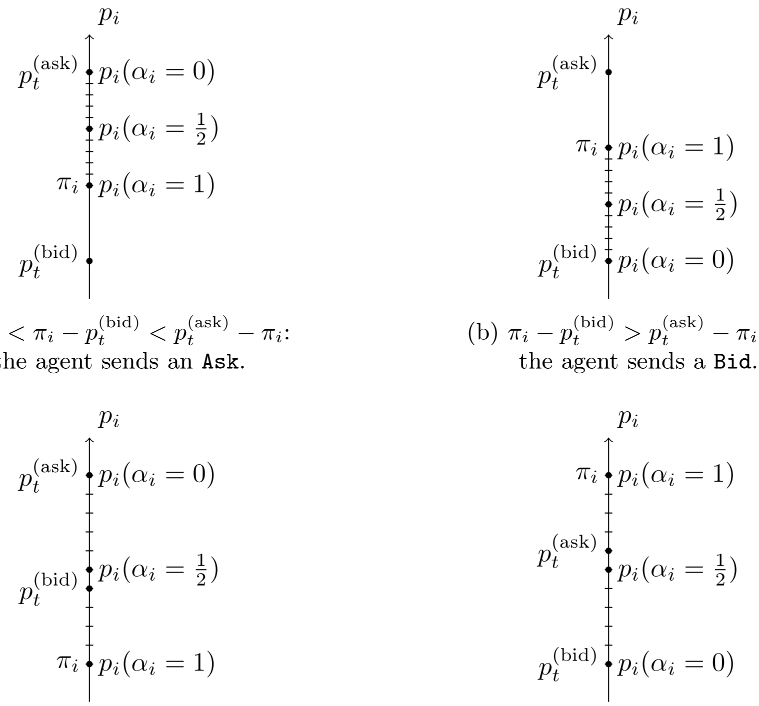
At time t, the order sent by this agent, which depends on the three previous parameters, is noted $o^{(f)}_{i,t}$. In addition, $p^{(ask)}_t$ and $p^{(bid)}_t$ represent the bestAsk and bestBid price in the order book, if there is at least an Ask and Bid in this order book, respectively; Otherwise, this quantity is equal to the last fixed price,[4] p_t. Then:

$$o^{(f)}_{i,t} = \begin{cases} (\texttt{Bid},\ p^{(ask)}_t,\ \kappa_i) & \text{if there is a least an Ask in the order book and if } p^{(ask)}_t \leq \pi_i \\ (\texttt{Ask},\ p^{(bid)}_t,\ \kappa_i) & \text{if there is a least a Bid in the order book and if } p^{(bid)}_t \geq \pi_i \\ (\texttt{Bid},\ p^{(bid)}_t + \alpha_i(\pi_i - p^{(bid)}_t),\ \kappa_i) & \text{otherwise, if } \pi_i - p^{(bid)}_t > p^{(ask)}_t - \pi_i \\ (\texttt{Ask},\ p^{(ask)}_t + \alpha_i(\pi_i - p^{(ask)}_t),\ \kappa_i) & \text{otherwise, if } \pi_i - p^{(bid)}_t < p^{(ask)}_t - \pi_i \\ \emptyset & \text{otherwise} \end{cases}$$

This order can be explained in the following way: the agent never buys at a price higher than π_i, nor sells at a lower price. If there already is an order in the order book that can be matched while respecting this constraint, it does so. Otherwise, it reduces the bid-ask spread by moving either the bestAsk or bestBid towards π_i while respecting the previous constraint, as shown in Fig. 4. The higher its aggressiveness α_i is, the more the agent tends to buy at a high price and sell at a low price.

Note that when the second situation arises, if its estimation of the fundamental value π_i is at equal distance from $p^{(ask)}_t$ and $p^{(bid)}_t$, then the agent does not send any order. It would be possible to introduce a bias to the agent, constant over

[4] There is always an opening price set in all our experiences. Thus, there is always a last fixed price.



(a) $0 < \pi_i - p_t^{(\text{bid})} < p_t^{(\text{ask})} - \pi_i$: the agent sends an `Ask`.

(b) $\pi_i - p_t^{(\text{bid})} > p_t^{(\text{ask})} - \pi_i > 0$: the agent sends a `Bid`.

(c) $\pi_i - p_t^{(\text{bid})} < 0 < p_t^{(\text{ask})} - \pi_i$ and $p_t^{(\text{bid})} < p_t^{(\text{ask})}$: the agent sends an `Ask`.

(d) $\pi_i - p_t^{(\text{bid})} > 0 > p_t^{(\text{ask})} - \pi_i$ and $p_t^{(\text{bid})} < p_t^{(\text{ask})}$: the agent sends a `Bid`.

Fig. 4. Price p_i of the order sent by the fundamentalist as a function of $p_t^{(\text{bid})}$, π_i, $p_t^{(\text{ask})}$ and α_i. The 4a and 4b situations represent the case where the estimation of the fundamental value is greater than the `bestBid` and smaller than the `bestAsk`. Figs. 4c and 4d represent what happens when π_i is smaller than p_t^{bid} or greater than p_t^{ask}, respectively. Note that four other situations can occur from time to time, when at least one of the `Bids` or `Asks` order book is empty, in which case $p_t^{\text{bid}} = p_t$ or $p_t^{\text{ask}} = p_t$. Under this assumption, one can have $\pi_i - p_t^{(\text{bid})} < p_t^{(\text{ask})} - \pi_i \leq 0$, $0 \geq \pi_i - p_t^{(\text{bid})} > p_t^{(\text{ask})} - \pi_i$ or either case 4c or 4d with $p_t^{(\text{bid})} \geq p_t^{(\text{ask})}$.

time, which would indicate whether it should buy or sell in this situation. However, this would only complicate the agent while not being useful to obtain a behaviour with which prices are continuously fixed.

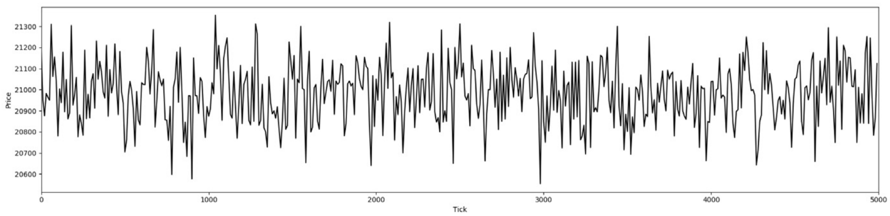
Finally, let us note that, given an agent, the quantity o_q of its orders o is constant, equal to κ_i. Despite being unrealistic, this way of determining the quantity makes it possible to have variable quantities, and thus, as discussed in Section 6.2, to be closer to the stylized facts.

3.2. Stylized facts

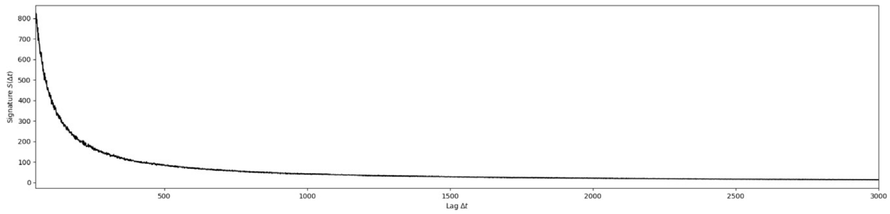
The following plots are obtained on a market on which 2200 agents are trading: there is one agent for each value of $(\alpha, \kappa, \pi) \in \{0, 0.1, \ldots, 1\} \times [\![1, 10]\!] \times \{20\,500, 20\,550, \ldots, 21\,400, 21\,450\}$ – which has a cardinality of $11 \cdot 10 \cdot 20 = 2\,200$. The simulation runs over 5000 ticks, where a tick consists in allowing each agent to talk once, in a fair random order. At the end of the simulation, 9 989 760 prices were fixed.

Fig. 5a represents the sequence of prices at the end of each tick, alongside with, on Fig. 5b, its signature plot. As expected, prices are continuously fixed. However, the behaviour of `DAT-F` being (too) simple, both plots are unrealistic: prices do not appear to be following a random walk, and the signature plot confirms this observation, as it converges towards 0.

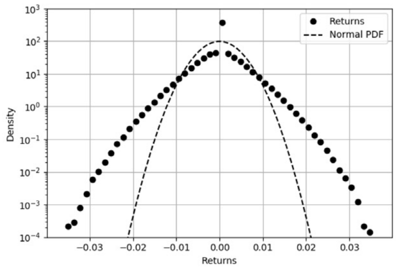
On the other end, Figs. 5c to 5f show that the unpretentious behaviour of `DAT-F` is enough to observe the stylized facts that focus on returns: Fig. 5c represents the distribution of logarithmic returns, whose kurtosis is equal to 7.89, compared to the distribution of the normal distribution that has the same mean and standard deviation: fat tails are observed and the distribution obtained is leptokurtic, since its kurtosis is higher than 3. Fig. 5d illustrates what is called in [21] "aggregational gaussianity", as it represents the distribution of logarithmic returns for a large timespan, i.e. the distribution of $(\log(p_{t+\Delta t}) - \log(p_t))_t$ for $\Delta t = 10^5$. This distribution is much closer to that of the normal distribution having the same mean and standard deviation, and has a kurtosis equal to 2.82, while the normal distribution has a kurtosis of 3. Note that this value of Δt roughly corresponds to 50 ticks, as, in average, 1998 prices are fixed during each



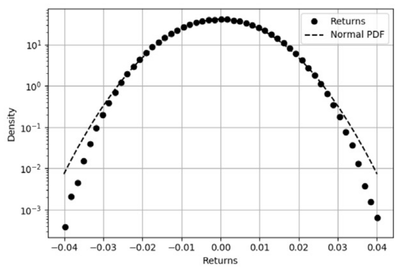
(a) Series of prices at the end of each tick.



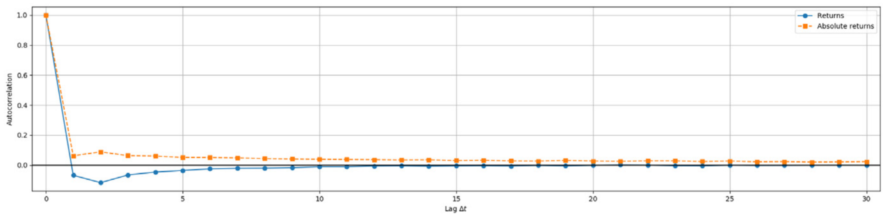
(b) Signature plot for the series of prices at the end of each tick.



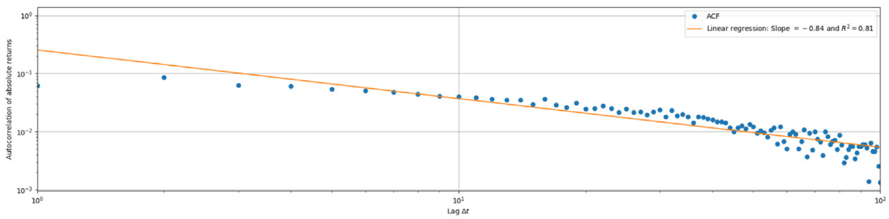
(c) Logarithmic returns distribution $(\log(p_{t+1}) - \log(p_t))_t$ (circles) compared to the normal distribution having the same mean and standard deviation (dashed line), on a semi-logarithmic scale.



(d) Logarithmic returns distribution $(\log(p_{t+\Delta t}) - \log(p_t))_t$ (circles) compared to the normal distribution having the same mean and standard deviation (dashed line) for a $\Delta t = 10^5$ lag, on a semi-logarithmic scale.



(e) Autocorrelation of logarithmic returns (solid blue line) and absolute values of logarithmic returns (dashed orange line).



(f) Autocorrelation of absolute values of logarithmic returns on a log-log scale.

Fig. 5. Results of a simulation using 2200 fundamentalist agents DAT-F during 5000 ticks.

tick. Finally, Figs. 5e and 5f show the autocorrelation function (acf) of returns and absolute returns: as expected, the acf of absolute returns decays as a power law. However, the exponent of the law is −0.81 and does not fall in the interval of values observed on real markets (between −0.30 and −0.20).

Thus, this agent allows us to simulate a market that respects some stylized facts that were previously described; While being simple, in the sense that it has few parameters, the behaviour of DAT-F illustrates that deterministic behaviours can lead to a market that does not freeze, while respecting some stylized facts. Thus, the following section aims at slightly refining our agent in order to get a market that reproduces stylized facts.

4. An informed and deterministic fundamentalist agent: DAT-IF

The informed fundamentalist behaviour proposed in this section extends the previous behaviour to constitute what we call a DAT-IF. These agents have an additional parameter, $\iota_i \in [\![0, 9]\!]$, which represents their level of information, and their estimation of the fundamental value now depends on time, and is therefore now noted $\pi_{i,t}$. This fundamental value is calculated in the same way as what is proposed in [22]: agent i has an estimation of the dividends paid by the company at the end of the future ι_i periods, that are subsequently called "days". From this estimation of future dividends comes the estimation of the fundamental value by the agent, which, within a given period, does not change. Within a given day, the behaviour of such DAT-IF is thus perfectly identical to the DAT-F behaviour described in the previous section.

4.1. Formalization of the extra-day behaviour

Let us note n_d the number of days in the simulation, $(d_j)_{1\leq j\leq n_d}$ the dividends paid at the end of the various days, $I = \max_i \iota_i + 1$ and T_j the set of moments making up the day j. One could note that a distinction is made between a moment and a tick: within the same tick, each agent talks, so several prices are set at different moments.

At the beginning of day j, agent i is given estimations $\widehat{d_j^{(j,i)}}, \dots, \widehat{d_{j+\iota_i-1}^{(j,i)}}$ of the future ι_i dividends. Each $\widehat{d_{j+k}^{(j,i)}}$ – the estimation of agent i at day j of the dividend d_{j+k} paid at end of day $j+k$ – , for $k \in [\![0, \iota_i - 1]\!]$ is uniformly chosen between $(1 - \varepsilon_{i,k})d_{j+k}$ and $(1 + \varepsilon_{i,k})d_{j+k}$, where $\varepsilon_{i,k} = \sqrt{k+1}\frac{I-\iota_i}{300}$: if a DAT-IF has a higher information level, then its estimation of the dividend will be, in average, closer to the real value. Likewise, the further the day for which an agent is given an estimation of the dividend is to today, the further the estimation will be to the real value of the dividend.

Now that the estimations of future dividends are defined, one can define the estimation of the fundamental value for DAT-IF agents who have a non-zero level of information, so for whom $\iota_i \geq 1$. This is done by using a generalization of Gordon's formula[5] [23]:

$$\forall j \in [\![1,\ n_d]\!], \forall t \in T_j,\ \pi_{i,t} = \sum_{k=0}^{\iota_i - 2} \frac{\widehat{d_{j+k}^{(j,i)}}}{(1+r_e)^k} + \frac{\widehat{d_{j+\iota_i-1}^{(j,i)}}}{r_e(1+r_e)^{\iota_i-2}}$$

where $r_e = 0.05$ is the risk adjusted interest rate.

For DAT-IF that have an information level ι_i equal to 0, we chose to take $\forall j \in [\![1,\ n_d]\!], \forall t \in T_j,\ \pi_{i,t} = p_t$, where p_t denotes the last fixed price at moment t. Unlike the informed fundamentalist agents in [22], the DAT-IF agents that have no information are not ZITs, but deterministic agents similar in every respect to the other informed agents. They just use the last price as their fundamental value, and therefore assume that the market is efficient. The second notable difference is that our DAT-IF agents do not have the exact value of future dividends but only an approximation of it.

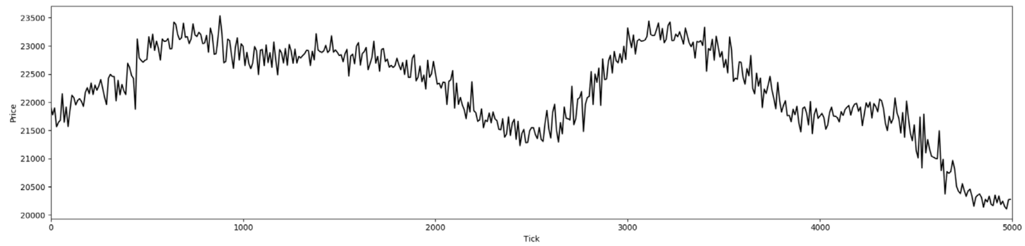
Finally, let us note that the intra-day behaviour of the agents of Toth et al. is entirely stochastic, in contrast to the behaviour proposed in 3.1. In the following simulations, $d_1 = 1000$ and d_{k+1} follows a normal law centred in d_k with a standard deviation of 10.

4.2. Stylized facts

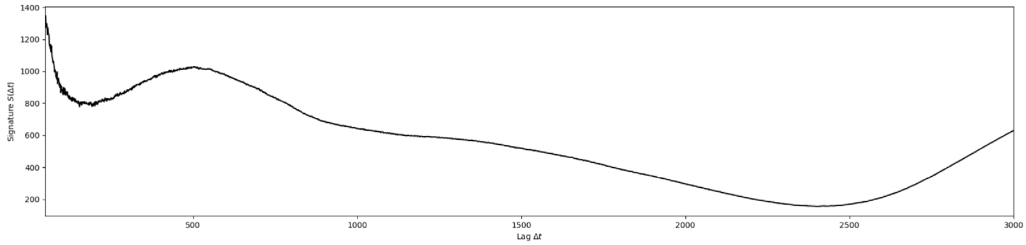
The following curves are obtained with a market where 2200 agents are trading: there are two agents for every $(\alpha, \iota, \kappa) \in \{0, 0.1, \dots, 1\} \times [\![0, 9]\!] \times [\![1, 10]\!]$ – set of cardinality $11 \cdot 10 \cdot 10 = 1100$. The opening price is set at 21 000, and each agent is initialized with the same amount of cash and the same amount of asset. The simulation runs during 100 days of 50 ticks, where, as a reminder, a tick consists of each agent having the possibility to send an order in a fair random order. At the end of the simulation, 9 996 531 prices were fixed.

Fig. 6 shows the series of prices at the end of each tick, and its signature plot. The price series looks like a random walk, unlike the one that was obtained using DAT-F agents: there are some small variation of prices on a short time interval, but there are also larger variations on longer intervals. The signature plot confirms it, as it does not converge

5 The original formula gives that the share price at time 0 is equal to $\sum_{t=0}^{\infty} \frac{D_t}{(1+r_e)^t}$ where D_t is the dividend that is expected at time t. The formula we use comes from taking $D_t = \widehat{d_{j+k}^{(j,i)}}$ for $t \leq \iota_i - 2$ and $D_t = \widehat{d_{j+\iota_i-1}^{(j,i)}}$ for $t \geq \iota_i - 1$ (for $t \geq \iota_i$, by definition of ι_i, agent i does not have any estimation of D_t, so it takes the last known estimation, which is $\widehat{d_{j+\iota_i-1}^{(j,i)}}$).



(a) Series of prices at the end of each tick.



(b) Signature plot for the series of prices at the end of each tick.

Fig. 6. Results of a simulation using 2200 informed fundamentalist agents DAT-IF during 100 days of 50 ticks: stylized facts using prices at the end of each tick.

towards 0 and is quite similar to the signature plot obtained using Microsoft shares – Fig. 2. Values for $\Delta t < 50$ are not represented as they are slightly – about five times – higher than those obtained for $\Delta t \geq 50$. This does not, in any way, change the fact that the signature plot has a strictly positive lower bound and a finite upper bound, like the signature plot of MS data.

The other stylized facts can be seen on Fig. 7: like in the market obtained using DAT-F agents, stylized facts focusing on returns can be observed. The only major difference is the value of the kurtosis, that can be found on Fig. 7c, alongside with the mean, standard deviation and skewness of the returns for our simulation, but also for the original simulation of [22] and the data they compared it to, which comes from General Electric stock. Reasons why the kurtosis is higher when DAT-IF are used rather than DAT-F, are briefly discussed in Section 6.2.

4.3. Evolution of agents' wealth

Agent-based simulations allow both a macroscopic study of general phenomena and a microscopic study of the evolution of each agent. In the same simulation as before, we no longer look at the market as a whole, but rather at individual agents. More specifically, by letting w_i^{init} and w_i^{final} be the initial wealth and final wealth of agent i, we are interested in the evolution of the agent's wealth, $\frac{w_i^{\text{final}} - w_i^{\text{init}}}{w_i^{\text{init}}}$, which is called return of agent i and noted r_i.

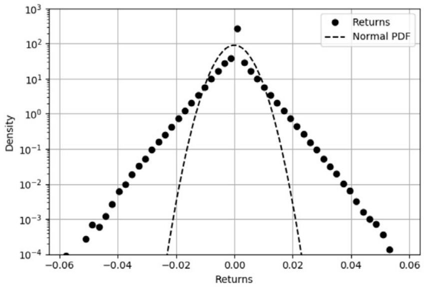
Fig. 8 presents the average of the returns of the agents according to their level of information, on a given simulation. By noting $R_\iota = \{r_i \mid \iota_i = \iota\}$ the set of returns of the agents having a level of information $\iota \in [\![0,\ 9]\!]$, each point corresponds to $(\iota,\ \mu(R_\iota))$, where μ denotes the empirical average. Lower and upper bounds of this mean are also shown using the standard deviation $\sigma(R_\iota)$ of R_ι.

The data obtained for average returns are similar to those obtained by [22]: on the one hand, it is better not to have any information at all than having little information and, on the other hand, highly informed agents get richer at the expense of less informed agents.

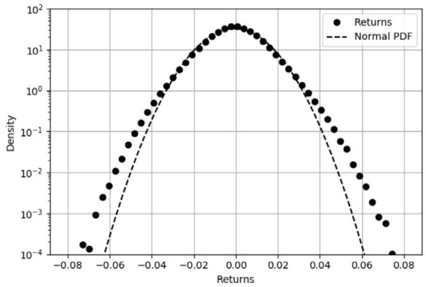
Thus, on a market with only DAT-IF agents, all stylized facts that were described in Section 2 can be observed, even the non-convergence of the signature plot. Moreover, by changing their information level ι, one can influence the direction towards which their wealth will evolve. However, in order to get an agent that may be lucky, we add a new parameter to our agent: according to the value of this parameter, randomness will highly or lightly influence the return of our Deterministic Artificial Trader.

5. A deterministic fundamentalist and chartist agent: DAT-IFC

The agent of Section 4 makes it possible to obtain a market respecting the stylized facts, while making it possible to control the probability that this agent gets richer through its level of information. However, it does not allow one to



(a) Logarithmic returns distribution $(\log(p_{t+1}) - \log(p_t))_t$ (circles) compared to the normal distribution having the same mean and standard deviation (dashed line), on a semi-logarithmic scale.



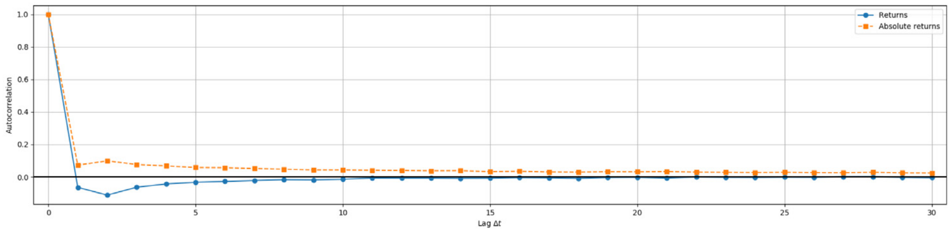
(b) Logarithmic returns distribution $(\log(p_{t+\Delta t}) - \log(p_t))_t$ (circles) compared to the normal distribution having the same mean and standard deviation (dashed line) for a $\Delta t = 10^5$ lag, on a semi-logarithmic scale.

	Simulation of Toth *et al.*	GE data	Our simulation
Mean	$3.52 \cdot 10^{-5}$	$2.12 \cdot 10^{-6}$	$-8.89 \cdot 10^{-9}$
St. deviation	$2.50 \cdot 10^{-2}$	$4.01 \cdot 10^{-4}$	$4.39 \cdot 10^{-3}$
Skewness	0.123	$-6.98 \cdot 10^{-2}$	$9.89 \cdot 10^{-4}$
Kurtosis	7.95	36.3	11.5

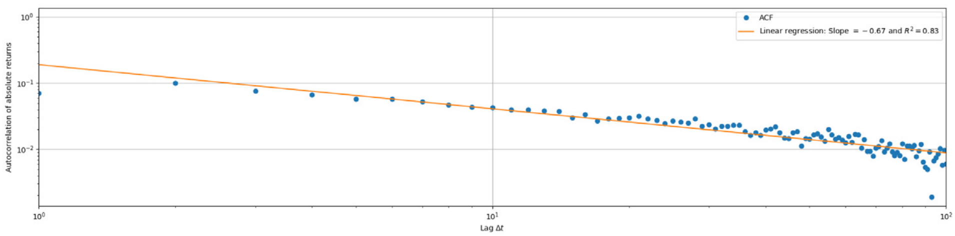
(c) Four first moments of logarithmic returns $(\log(p_{t+1}) - \log(p_t))_t$ for our simulation, that of Toth *et al.* and for General Electric (GE) data.

	Our simulation	Normal law
Mean	$-6.97 \cdot 10^{-4}$	
St. deviation	$1.23 \cdot 10^{-2}$	
Skewness	$1.33 \cdot 10^{-2}$	0
Kurtosis	3.76	3

(d) The first four moments of logarithmic returns $(\log(p_{t+\Delta t}) - \log(p_t))_t$ for $\Delta t = 10^5$ and the distribution of the normal law having the same mean and standard deviation.



(e) Autocorrelation of logarithmic returns (solid blue line) and absolute values of logarithmic returns (dashed orange line).



(f) Autocorrelation of absolute values of logarithmic returns on a log-log scale.

Fig. 7. Results of a simulation using 2200 informed fundamentalist agents DAT−IF during 100 days of 50 ticks: stylized facts using all prices.

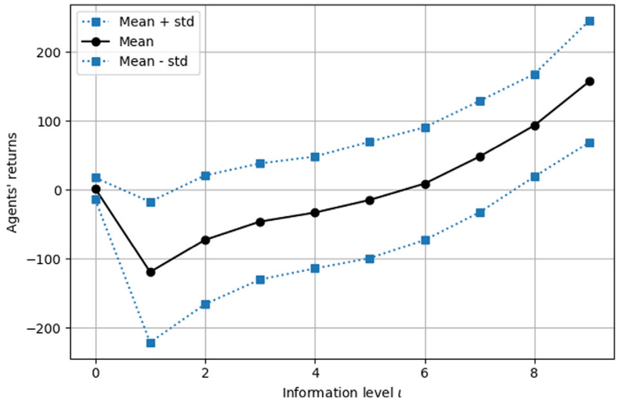


Fig. 8. Values obtained, on a single simulation, of the average returns of agents according to their level of information $(\iota,\ \mu(R_\iota))_{\iota\in[\![0,\ 9]\!]}$, alongside with a lower bound $(\iota,\ \mu(R_\iota)-\sigma(R_\iota))_{\iota\in[\![0,\ 9]\!]}$ and an upper bound $(\iota,\ \mu(R_\iota)+\sigma(R_\iota))_{\iota\in[\![0,\ 9]\!]}$ of this mean value.

observe the desired result, which is to have rich agents either because they are either well-informed, or because they are lucky. To obtain a market verifying this constraint, it is thus necessary to add another behaviour: here, a chartist with sliding average will be used. There are two ways to proceed: either adding agents with this second behaviour, or modifying our agents so that their behaviour is a weighted average of the fundamentalist behaviour described in Section 4 and the new behaviour. We chose the second method, in a concern for realism: a trader is not destined to be only fundamentalist or chartist: it can take both its estimation of the fundamental value and the price trend into account to decide which order it will send. The resulting agent relying on both an informed fundamentalist behaviour and a chartist behaviour is called `DAT-IFC`.

The difficulty of this task lies in defining a weighted average of behaviours, without losing the deterministic character of the behaviour. Note that if no importance were given to determinism, it would be sufficient to define this mean behaviour by sending the order determined by the first behaviour with a certain probability, and to send the order determined by the second behaviour otherwise.

5.1. Weighted average of deterministic behaviours

A behaviour is a function that takes as input the market state and the agent state, and that returns an order o that is either a limit order or the null order ∅. Thus, in order to construct a behaviour that implements a weighted average of the informed fundamentalist strategy `DAT-IF` and of the chartist strategy that is defined in Section 5.2, we propose to define a weighted average of behaviours as the function that takes the same input as those behaviours and returns the weighted average of the orders returned by those behaviours.

Then, we define the average of the orders $(o_k)_{1\le k\le n}$ weighted by[6] $(\lambda_k)_{1\le k\le n}$, denoted $\overline{(o_k,\lambda_k)_{1\le k\le n}}$, in the following way: if o_k is a limit order, then let d_k be its direction and p_k its price ; otherwise, if o_k is the null order ∅, let $d_k = p_k = \emptyset$. A value, called numerical direction, is assigned to each direction : $g(\texttt{Ask}) = +1$, $g(\texttt{Bid}) = -1$ and $g(\emptyset) = 0$. It is used to compute the weighted average of the numerical directions $d = \sum_{k=1}^{n} \lambda_k g(d_k)$ and the averaged direction, denoted $\tilde{d}$ and defined by:

$$\tilde{d} = \begin{cases} \texttt{Ask} & \text{if } d > 0 \\ \emptyset & \text{if } d = 0 \\ \texttt{Bid} & \text{if } d < 0 \end{cases}$$

This direction will be the direction of the averaged order. Then let K be the set of indexes indexing an order whose direction is equal to the averaged direction, i.e.: $K = \{k \in [\![1,\ n]\!] \mid d_k = \tilde{d}\}$, and let $\Lambda = \sum_{k=K} \lambda_k$ be the total weight of the orders indexed by K.

Finally, let the resulting order be defined by:

$$\overline{(o_k,\lambda_k)_{1\le k\le n}} = \begin{cases} \emptyset & \text{if } \tilde{d} = \emptyset \\ \left(\tilde{d},\ \sum_{k\in K} \dfrac{\lambda_k}{\Lambda} p_k,\ \lceil |d| q_0 \rceil\right) & \text{otherwise} \end{cases}$$

[6] By definition, $\sum_{k=1}^{n} \lambda_k = 1$.

where $\lceil\cdot\rceil$ is the ceiling function and q_0 is the maximal quantity of an order that an agent can send – in the following simulations, $q_0 = 10$.

The formalization of this definition does not reveal the elementary simplicity of the idea it formalizes, which can be summarized by:

- The averaged direction $\tilde{d}$ is the predominant direction (with weights being taken into account). If there is no predominant direction, the order returned is the null order Ø.
- The price of the averaged order is the weighted average of the prices of the different orders whose direction is equal to the averaged direction.
- The averaged quantity is $\lceil |d| q_0 \rceil$, and is chosen so that it reaches its maximum value when every order o_k, with $1 \le k \le n$, have the same direction.

Note that the quantities of orders o_k have no influence on the resulting order. We made this choice because it allows us not to have to take into account the quantity of the order returned by the fundamentalist behaviour, which was set by a parameter κ_i of self-confidence, which was arbitrary and intrinsic to the DAT-F and DAT-IF agents. By doing so, it is therefore possible to do without this parameter, which minimizes the number of parameters characterizing the agents. Let us also note that many behaviours that were proposed in the literature settle for sending orders whose quantity is equal to 1, because it is often difficult to define a quantity that makes sense. The method proposed here makes it possible to circumvent this difficulty: it is enough to define so-called elementary behaviours, without defining the quantity of the orders sent by those behaviours. Then, by applying this weighted average, a quantity, which is neither arbitrary nor always equal to 1, is defined.[7] In the following example, the quantities of the o_1, o_2 and o_3 orders have therefore not been specified, and in accordance with the notation stated in the introduction, have been replaced by the _ character.

Finally, note that this weighted average of orders have the properties that one can expect from an average:

- If each o_k, for $k \in [\![1,\ n]\!]$, has the same direction d_0 and price p_0, then the averaged direction and price will be d_0 and p_0, respectively.
- The averaged price is greater or equal to $\min_k p_k$ and smaller or equal to $\max_k p_k$
- If for some k, p_k is non-decreasing, then the averaged price is also non-decreasing.

Example. Let $q_0 = 10$, and consider three orders:

1. $o_1 = (\texttt{Ask}, 1000, \_)$ of weight $\frac{1}{2}$
2. $o_2 = (\texttt{Ask}, 1100, \_)$ of weight $\frac{1}{6}$
3. $o_3 = (\texttt{Bid}, 950, \_)$ of weight $\frac{1}{3}$

Then $d = \frac{1}{2} + \frac{1}{6} - \frac{1}{3} = \frac{1}{3}$ and thus $\tilde{d} = \texttt{Ask}$. As $\frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{6}} \cdot 1000 + \frac{\frac{1}{6}}{\frac{1}{2}+\frac{1}{6}} \cdot 1100 = 1025$, there comes $\overline{(o_k, \lambda_k)_{1 \le k \le 3}} = (\texttt{Ask},\ 1025,\ 4)$ as $\lceil \frac{10}{3} \rceil = 4$.

5.2. Formalization of the behaviour

The average of several behaviours being defined, it only remains to specify the chartist behaviour for our DAT-IFC agent to be totally defined. The intrinsic parameters of agent i are:

- $\alpha_i \in [0, 1]$: its aggressiveness
- $\iota_i \in \mathbb{N}$: its information level
- $\pi_{i,t} \in \mathbb{N}^*$: its estimation of the fundamental value
- $\tau_i \in [0, 1]$: the importance it gives to price trends in its decision-making
- $\rho_i \in \{-1, 1\}$: what it thinks about price trends: if $\rho_i = 1$, it thinks that trends tend to persist (if the price has previously increased, then it will probably continue to increase), and if $\rho_i = -1$, it thinks trends tend to reverse.

On the one hand, with the parameters α_i and $\pi_{i,t}$ – $\pi_{i,t}$ being defined thanks to ι_i – [8], the agent determines which order $o^{(f)}_{i,t}$ it would have sent if it was purely fundamentalist, given by the behaviour described in Section 4. On the other hand, it determines what is the order it would have sent if it was purely chartist, noted $o^{(c)}_{i,t}$ and defined in the following paragraphs. This order depends on α_i, on ρ_i and on the market at time t. The order that the agent will ultimately send is the resulting order $o_{i,t} = \overline{(o^{(f)}_{i,t},\ 1 - \tau_i),\ (o^{(c)}_{i,t},\ \tau_i)}$: it is a mean between the two previous orders, such that the weight of the "chartist order" $o^{(c)}_{i,t}$ is τ_i – and thus the weight of the "fundamentalist order" $o^{(f)}_{i,t}$ is $1 - \tau_i$. The extra-day behaviour is identical to what it described in the previous section.

7 However, if we want the quantities q_k of the orders o_k to have an influence on the resulting order, it is possible to define the resulting quantity in a similar way to the price, by defining it as $\sum_{k \in K} \frac{\lambda_k}{\Lambda} q_k$.

8 Unlike other parameters, $\pi_{i,t}$ is not a parameter given to an agent when it is created, but a parameter that is calculated at the beginning of each day.

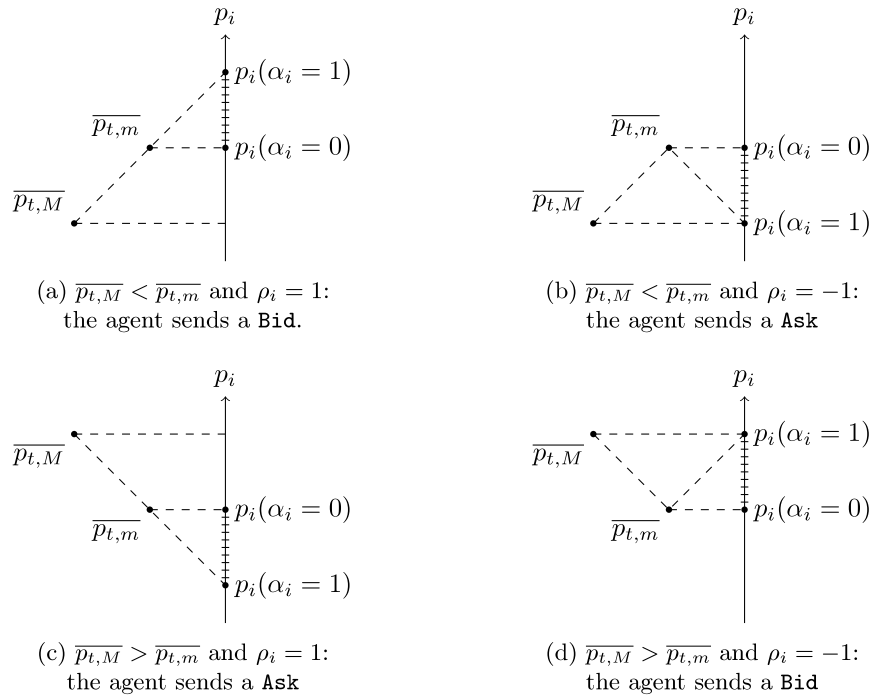


Fig. 9. Price p_i of the order sent by the chartist agent as a function of $\overline{p_{t,M}}$, $\overline{p_{t,m}}$, α_i and ρ_i. Only the extremes values of p_i – for $\alpha_i = 0$ and $\alpha_i = 1$ – are represented.

The chartist behaviour with moving average used by the DAT-IFC is defined as follows: For $T > 0$, if p_j now denotes the last price fixed at tick j, let $\overline{p_{t,T}} = \frac{1}{T}\sum_{j=t-T+1}^{t} p_j$ be the average of the prices at the end of the last T ticks, if it can be defined, and $\emptyset$ otherwise, i.e. if the price history does not include enough prices ($t < T - 1$).

Let m and M two integers such that $m < M$. In the following simulations, we took $m = 20$ and $M = 50$. Then:

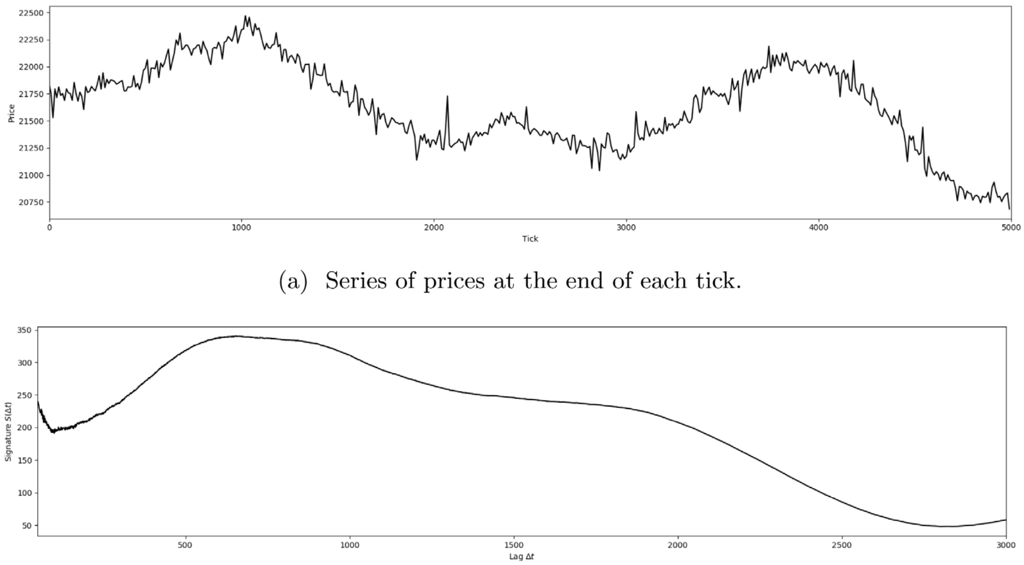
$$o_{i,t}^{(c)} = \begin{cases} (\texttt{Bid},\ \overline{p_{t,m}} + \alpha_i|\overline{p_{t,m}} - \overline{p_{t,M}}|,\ \_) & \text{if } \overline{p_{t,M}} \neq \emptyset \text{ and } [(\rho_i = 1 \text{ and } \overline{p_{t,m}} > \overline{p_{t,M}}) \\ & \text{or } (\rho_i = -1 \text{ and } \overline{p_{t,m}} < \overline{p_{t,M}})] \\ (\texttt{Ask},\ \overline{p_{t,m}} - \alpha_i|\overline{p_{t,m}} - \overline{p_{t,M}}|,\ \_) & \text{if } \overline{p_{t,M}} \neq \emptyset \text{ and } [(\rho_i = 1 \text{ and } \overline{p_{t,m}} < \overline{p_{t,M}}) \\ & \text{or } (\rho_i = -1 \text{ and } \overline{p_{t,m}} > \overline{p_{t,M}})] \\ \emptyset & \text{otherwise, i.e. if } \overline{p_{t,M}} = \emptyset \text{ or } \overline{p_{t,m}} = \overline{p_{t,M}} \end{cases}$$

Less formally, the agent does not send any order if the price history contains less than M price, or if the average of the last m prices is equal to the average of the last M prices. Otherwise, if the agent thinks that trends are persistent and if prices have increased ($\overline{p_{t,m}} > \overline{p_{t,M}}$), then by noting δ the difference between those prices, the agent will send an order Bid at a price set between $\overline{p_{t,m}}$ and $\overline{p_{t,m}} + \delta$. In the same situation, if the agent thinks that trends tends to reverse, then it will send an order between $\overline{p_{t,m}}$ and $\overline{p_{t,m}} - \delta$. Cases when prices have decreased are symmetrical. This is summarized in Fig. 9.

5.3. Stylized facts

The following curves are obtained with a market on which 2200 agents interact: there is one agent for each value of $(\alpha, \iota, \tau, \rho) \in \{0,\ 0.1,\ \ldots,\ 1\} \times [\![0,\ 9]\!] \times \{0,\ \frac{1}{9},\ \ldots,\ 1\} \times \{-1, 1\}$ – of cardinality $11 \cdot 10 \cdot 10 \cdot 2 = 2\,200$. The opening price is set at 21 000. The simulation lasts 100 days of 50 ticks.

At the end of the simulation, 9 142 968 prices were fixed. As before, Fig. 10 shows the price series and its signature plots, while Fig. 11 presents the data related to logarithmic returns. There is no significant difference with the data that was obtained using DAT-IF agents, apart from the kurtosis of returns which is significantly higher than the ones from Sections 3 and 4, and the exponent of the power law followed by the acf of absolute returns that falls within the range given in [21].



(a) Series of prices at the end of each tick.

(b) Signature plot for the series of prices at the end of each tick.

Fig. 10. Results of a simulation using 2200 DAT-IFC agents, that are both informed fundamentalists and chartists, during 100 days of 50 ticks: stylized facts using prices at the end of each tick.

5.4. Evolution of agents' wealth

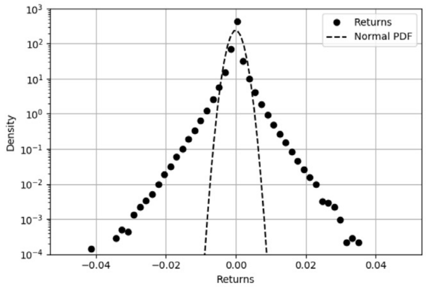
Keeping the same definition of an agent's return as previously, Fig. 12 presents the average returns of agents according to their level of information and their proportion of chartist behaviour. Let us note $R_{\iota,\ \tau} = \{r_i \mid \iota_i = \iota \wedge \tau_i = \tau\}$ the set of returns of agents having an information level equal to ι and a proportion of chartist behaviour equal to τ.

For all the points such as $\tau = 0$, the curve that is found is similar to the curve of the returns for the informed fundamentalist agents DAT-IF – Fig. 8 – , which is at first hardly surprising: agents such that $\tau = 0$ are identical to the agents of the previous section. However, this means that their behaviour is "stable": despite the presence of agents with a different τ, very well-informed purely fundamentalist agents continue to have higher returns than poorly informed purely fundamentalist agents.

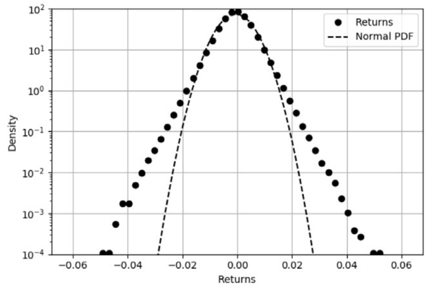
Moreover, the phenomenon that we wanted is emerging from this market: on the one hand, in average, purely fundamentalist agents with a high level of information are among the agents who have the highest returns; on the other hand, there are agents who are not well-informed but who are lucky and have returns as high as the agents who are very well informed. This is reflected in the fact that there exists $\iota \leq 4$ and τ such that $\mu(R_{\iota,\ \tau}) + \sigma(R_{\iota,\ \tau}) > \mu(R_{9,\ 0}) - \sigma(R_{9,\ 0})$. For example, $\mu(R_{0,\ 1}) + \sigma(R_{0,\ 1}) = 39.82$ and $\mu(R_{9,\ 0}) - \sigma(R_{9,\ 0}) = 38.11$. Hence, there is a non-negligible proportion of uninformed and purely chartist agents who have similar returns to those obtained by highly informed and purely fundamentalist agents.

This observation was made on every simulation that we made: we ran 100 simulations lasting 50 days of 50 ticks, and 66 of those were such that $\mu(R_{0,\ 1}) + \sigma(R_{0,\ 1}) \geq \mu(R_{9,\ 0}) - \sigma(R_{9,\ 0})$, and among the 34 other simulations, the relative difference between those two quantities was smaller than one tenth on 27 simulations. In the 7 other cases, this difference was always smaller than one fourth.

It is thus possible to conclude that the lucky and uninformed chartists among DAT-IFC have returns of the same order of magnitude as the highly informed agents, thus making it possible to arrive at a market respecting all the fundamental characteristics of an artificial market set out in the introduction: prices are continuously fixed, the obtained market respects the most commonly studied stylized facts – fat tails, aggregational gaussianity, absence of autocorrelation of returns, and decay of the acf of absolute returns as a power law – and there is a behavioural differentiation depending on the parameters of the agents. This differentiation leads to the fact that the most fundamentalist and well-informed agents are among the richest agents ; Nevertheless, there are also agents with little information who, being chartists and lucky, are also in the latter category. It means that the randomness that emerged at the market level gave rise to some randomness at the agents level.



(a) Logarithmic returns distribution $(\log(p_{t+1}) - \log(p_t))_t$ (circles) compared to the normal distribution having the same mean and standard deviation (dashed line), on a semi-logarithmic scale.



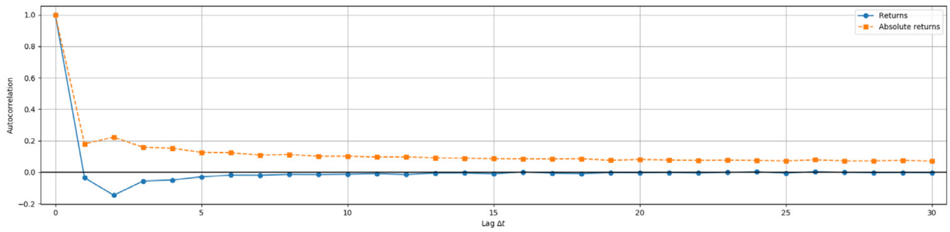
(b) Logarithmic returns distribution $(\log(p_{t+\Delta t}) - \log(p_t))_t$ (circles) compared to the normal distribution having the same mean and standard deviation (dashed line) for a $\Delta t = 10^5$ lag, on a semi-logarithmic scale.

	Simulation of Toth *et al.*	GE data	Our simulation
Mean	$3.52 \cdot 10^{-5}$	$2.12 \cdot 10^{-6}$	$-5.85 \cdot 10^{-9}$
St. deviation	$2.50 \cdot 10^{-2}$	$4.01 \cdot 10^{-4}$	$1.63 \cdot 10^{-3}$
Skewness	0.123	$-6.98 \cdot 10^{-2}$	$4.41 \cdot 10^{-3}$
Kurtosis	7.95	36.3	29.4

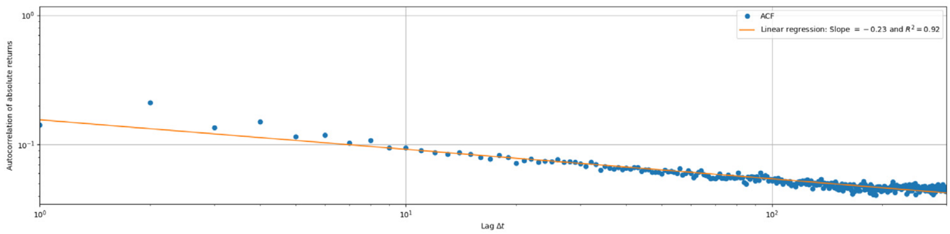
(c) Four first moments of logarithmic returns $(\log(p_{t+1}) - \log(p_t))_t$ for our simulation, that of Toth *et al.* and for General Electric (GE) data.

	Our simulation	Normal law
Mean	$-5.39 \cdot 10^{-4}$	
St. deviation	$5.44 \cdot 10^{-3}$	
Skewness	$-2.23 \cdot 10^{-2}$	0
Kurtosis	4.71	3

(d) The first four moments of logarithmic returns $(\log(p_{t+\Delta t}) - \log(p_t))_t$ for $\Delta t = 10^5$ and the distribution of the normal law having the same mean and standard deviation.

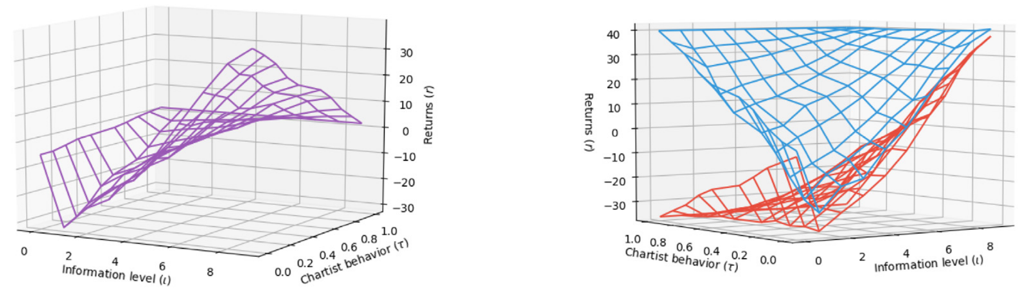


(e) Autocorrelation of logarithmic returns (solid blue line) and absolute values of logarithmic returns (dashed orange line).



(f) Autocorrelation of absolute values of logarithmic returns on a log-log scale.

Fig. 11. Results of a simulation using 2200 `DAT-IFC` agents that are both informed fundamentalists and chartists, during 100 days of 50 ticks: stylized facts using all prices.



(a) Returns average of agents for a given level of information and a proportion of chartist behaviour. Each point corresponds to a value of $(\iota,\ \tau,\ \mu(R_{\iota,\ \tau}))$.

(b) Framing of the returns of agents. Each point corresponds either to a value of $(\iota,\ \tau,\ \mu(R_{\iota,\ \tau}) - \sigma(R_{\iota,\ \tau}))$ (red points) or $(\iota,\ \tau,\ \mu(R_{\iota,\ \tau}) + \sigma(R_{\iota,\ \tau}))$ (blue points).

Fig. 12. Average returns – Fig. 12a – of the agents according to their parameters, and a lower and upper bound of average returns, using the standard deviation – Fig. 12b.

6. Discussion

6.1. Influence of parameters

This subsection aims at briefly explaining what would have happened if different parameters were chosen, for example if the estimations of the fundamental value $(\pi_i)_i$ were different for the DAT−F agents.

DAT-F A larger standard deviation of the $(\pi_i)_i$ results in a larger standard deviation of the fixed prices $(p_t)_t$, and a decrease or increase in the average value of the $(\pi_i)_i$ results in an equal decrease or increase in the average value of the fixed prices, respectively. In addition, if the set of possible values for the fundamental value is small, the distribution of returns is no longer continuous, but only piecewise continuous.[9] This phenomenon, although unusual, does not however call into question the four stylized facts described above: they remain present, even with a piecewise continuous distribution. For a set of fundamental values greater than ten, this phenomenon disappears. Moreover, increasing the number of values that can be taken by the aggressiveness α_i, parameter necessary to deterministically determine order prices, does not seem to have a significant influence on the results obtained. Finally, the influence of the values taken by self-confidence κ_i, i.e. the quantity of orders sent by agents, is discussed in Section 6.2.

DAT-IF The curve of average returns of agents as a function of their information level, depicted on Fig. 8, is conditioned by the number of days during which the simulation runs. If the latter is too small, statistical fluctuations in the sequence of dividends may cause the resulting curve not to conform to Fig. 8, but becomes decreasing on $\iota \in [\![1,\ 9]\!]$, when the dividend series is decreasing almost everywhere. If this deviant case seems to call into question this behaviour, choosing a large number of days in front of the maximum level of information makes it possible to avoid it, since the probability that the sequence of dividends is decreasing or almost decreasing is then negligible. Moreover, our behaviour is based on a generalization of Gordon's formula, derived from a model considering that the sequence of dividends is increasing and whose growth rate is constant. Thus, the fact that a deviant case appears when our behaviour is faced with a decreasing dividend stream is hardly surprising.

DAT-IFC Finally, let us note that in order to obtain a simulation such that $\mu(R_{0,\ 1}) + \sigma(R_{0,\ 1})$ and $\mu(R_{9,\ 0}) - \sigma(R_{9,\ 0})$ are close , it seems necessary to have a value of M – defined in 5.2 – close to the number of ticks in a day. In other words, lucky agents among purely chartist agents ($\tau = 1$) can do as well as highly informed fundamentalist agents ($\tau = 0$ and $\iota = 9$) only when they compare the average of the latest prices to the average of prices over a time interval equal to one day.

6.2. Perspectives

Origins of kurtosis On real financial markets, the kurtosis of the logarithmic returns has high values for small timespans: 36.3 for General Electric in [22], 15.95 for S&P 500 futures and 60 for the exchange rate dollar/Swiss franc

9 Obviously, if this set is reduced to a singleton $\{\pi\}$, then as all agents buy only at a price strictly below π and all agents sell only at a price strictly above π, no price is ever fixed.

in [21] with a timespan of 5 min. It also displays high values from larger timespans for some markets, such as the Bitcoin market, for which the kurtosis is between 27 and 108 with a timespan between 5 and 12 h [24]. Using DAT−IFC agents, and by running 100 simulations, we found values of the kurtosis whose average was 31.6. Those values are much closer to the values observed on a real market than those obtained with DAT−F and DAT−IF, but also than values found in [22] or [25] using double-auction markets: their kurtosis was approximately equal to 7. Why do DAT−IFC allows us to get price series whose returns have a kurtosis much closer to the one observed on real markets? We made the following observations:

- The maximum quantity send by the agents – which can be controlled by modifying the values of κ_i for DAT−F and DAT−IF agents, or by changing the value of q_0, defined in Section 5.1 for DAT−IFC – seems to greatly influence the value of the kurtosis: the higher the quantities, the higher the kurtosis. For example, a market with DAT−IF agents and $q_0 = 30$ yields an average kurtosis of 37.9 – recall that we got 31.6 when $q_0 = 10$.
- The more *heterogeneous* the behaviours are, the higher the kurtosis is: DAT−IFC agents give a higher kurtosis than DAT−IF traders, who also give a higher kurtosis than DAT−F.

The origins of the high values of the kurtosis on financial markets are complex and caused by numerous factors. For example, [26] shows that herd-like behaviours in financial markets lead to higher values for the kurtosis of the returns, reflecting the presence of particularly heavy tails. We think that a natural perspective of this paper would be a quantitative study of the influence of both the quantity that are part of the orders sent by agents and the heterogeneity of the behaviours of the agents on the value of the kurtosis. Of course, in order to do so, one would need a quantitative definition of the heterogeneity of agents, which, to our knowledge, has not yet been proposed in the literature.

Statistical learning Recall that our focus was the following question: "*Can randomness emerge at the macroscopic level from microscopic determinism on a double-auction order book, and can it, through the feedback loops that define complex systems, lead to some randomness at the microscopic level?*" Our DAT−IFC agent provides a positive answer to this question, and has the following interesting property: while agents that have a small value of τ_i and a high information level ι_i always get rich, agents that have a high value of τ_i can experience luck: some of them will get rich and some of them will get poor. This mean that while their behaviour is deterministic, the final wealth of those agents is random.[10] Thus, one interesting question would be if, using statistical learning tools, one can discriminate between agents that got rich because they had a lot of information, and those that got rich simply because they were lucky, just by looking at the orders sent by the agents. If this can be done on our model, it would be worth investigating applications to real markets, for example to detect fraudulent behaviours.

References

[1] Gew-rae Kim, Harry Markowitz, Investment rules, margin, and market volatility, J. Portfolio Manag. 16 (1) (1989) 45–52.
[2] Gerhard Weiss, Multiagent Systems: A Modern Approach to Distributed Artificial Intelligence, MIT press, 1999.
[3] Michael Wooldridge, An Introduction to Multiagent Systems, John Wiley & Sons, 2009.
[4] J. Doyne Farmer, Mauro Gallegati, Cars Hommes, Alan Kirman, Paul Ormerod, Silvano Cincotti, Anxo Sanchez, Dirk Helbing, A complex systems approach to constructing better models for managing financial markets and the economy, Eur. Phys. J. Spec. Top. 214 (1) (2012) 295–324.
[5] Thomas Lux, Michele Marchesi, Scaling and criticality in a stochastic multi-agent model of a financial market, Nature 397 (6719) (1999) 498.
[6] Johann Lussange, Alexis Belianin, Sacha Bourgeois-Gironde, Boris Gutkin, A bright future for financial agent-based models. arXiv preprint arXiv:1801.08222, 2018.
[7] Hans M. Amman, Leigh Tesfatsion, David A. Kendrick, Kenneth L. Judd, John Rust, Handbook of Computational Economics, Vol. 2, Elsevier, 1996.
[8] Philippe Mathieu, Olivier Brandouy, A generic architecture for realistic simulations of complex financial dynamics, in: Advances in Practical Applications of Agents and Multiagent Systems, Springer, 2010, pp. 185–197.
[9] Philippe Mathieu, Olivier Brandouy, Introducing atom, in: Advances on Practical Applications of Agents and Multi-Agent Systems, Springer, 2012, pp. 269–272.
[10] Richard G. Palmer, W. Brian Arthur, John Henry Holland, Blake LeBaron, Paul Tayler, Artificial Economic Life: A Simple Model of a Stockmarket, 1994.
[11] Blake LeBaron, Agent-based computational finance, Handb. Comput. Econ. 2 (2006) 1187–1233.
[12] Marco Raberto, Silvano Cincotti, Sergio M. Focardi, Michele Marchesi, Agent-based simulation of a financial market, Physica A 299 (1-2) (2001) 319–327.
[13] Dhananjay K. Gode, Shyam Sunder, Allocative efficiency of markets with zero-intelligence traders: Market as a partial substitute for individual rationality, J. Polit. Econ. 101 (1) (1993) 119–137.
[14] Ryuichi Yamamoto, Order aggressiveness, pre-trade transparency, and long memory in an order-driven market, J. Econom. Dynam. Control 35 (11) (2011) 1938–1963.
[15] J. Doyne Farmer, Paolo Patelli, Ilija I. Zovko, The predictive power of zero intelligence in financial markets, Proc. Natl. Acad. Sci. 102 (6) (2005) 2254–2259.
[16] Carl Chiarella, Giulia Iori, et al., A simulation analysis of the microstructure of double auction markets, Quant. Financ. 2 (5) (2002) 346–353.
[17] Shyam Sunder, Markets as Artifacts: Aggregate Efficiency from Zero-Intelligence Traders, 2003.
[18] Alan Kirman, Whom or what does the representative individual represent?, J. Econ. Perspect. 6 (1992) 117–136.
[19] Aki-Hiro Sato, Hideki Takayasu, Dynamic numerical models of stock market price: from microscopic determinism to macroscopic randomness, Physica A 250 (1) (1998) 231–252.

[10] These results illustrate, as B. Mandelbrot already said [27], that market finance is a complex system. We show that this complexity and randomness can arise from a single, simple and entirely deterministic behaviour, which is remarkable, and that this randomness can lead to some randomness at the agents' level.

[20] Richard Bookstaber, Alan Kirman, Modeling a Heterogeneous World, 2018.
[21] Rama Cont, Empirical properties of asset returns: stylized facts and statistical issues, Quant. Finance 1 (2) (2001) 223–236.
[22] Bence Toth, Enrico Scalas, Jürgen Huber, Michael Kirchler, The value of information in a multi-agent market model, Eur. Phys. J. B 55 (1) (2007) 115–120.
[23] Myron J. Gordon, Eli Shapiro, Capital equipment analysis: the required rate of profit, Manag. Sci. 3 (1) (1956) 102–110.
[24] Aurelio F. Bariviera, María José Basgall, Waldo Hasperué, Marcelo Naiouf, Some stylized facts of the bitcoin market, Physica A 484 (2017) 82–90.
[25] Francois Ghoulmie, Rama Cont, Jean-Pierre Nadal, Heterogeneity and feedback in an agent-based market model, J. Phys.: Condens. Matter 17 (14) (2005) S1259.
[26] Rama Cont, Jean-Philipe Bouchaud, Herd behavior and aggregate fluctuations in financial markets, Macroecon. Dyn. 4 (2) (2000) 170–196.
[27] Benoit B. Mandelbrot, Stochastic Volatility, Power Laws and Long Memory, 2001.