# Algebras for Regular Relations

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Work in progress!

Highlights '23
Kassel
27 July 2023

# Regular Relations

$R \subseteq \Sigma^* \times \Sigma^*$

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$$(aa\ ,\ abbb) \rightsquigarrow \begin{pmatrix} a & a & \overbrace{\Box \quad \Box}^{\text{padding}} \\ a & b & b \quad\;\; b \end{pmatrix}$$

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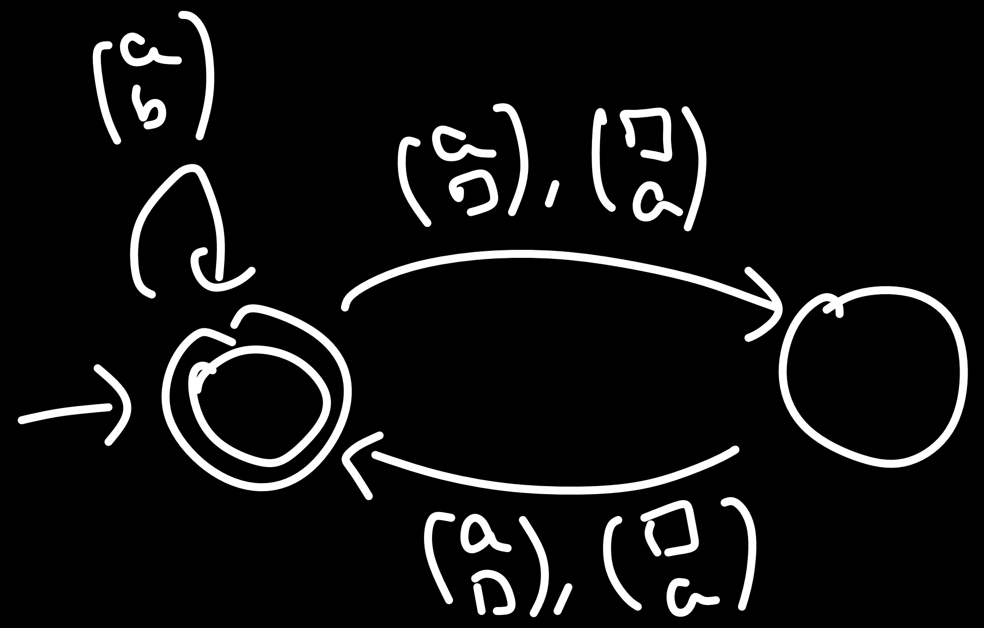
**Ex**




accepts
$\text{Same\_Parity} = \{(u,v) \mid |u| \equiv |v| \bmod 2\}$.

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$(aa\ ,\ abbb) \rightsquigarrow \begin{pmatrix} a & a & \overbrace{\square \quad \square} \\ a & b & b \quad\;\; b \end{pmatrix}$ ✓

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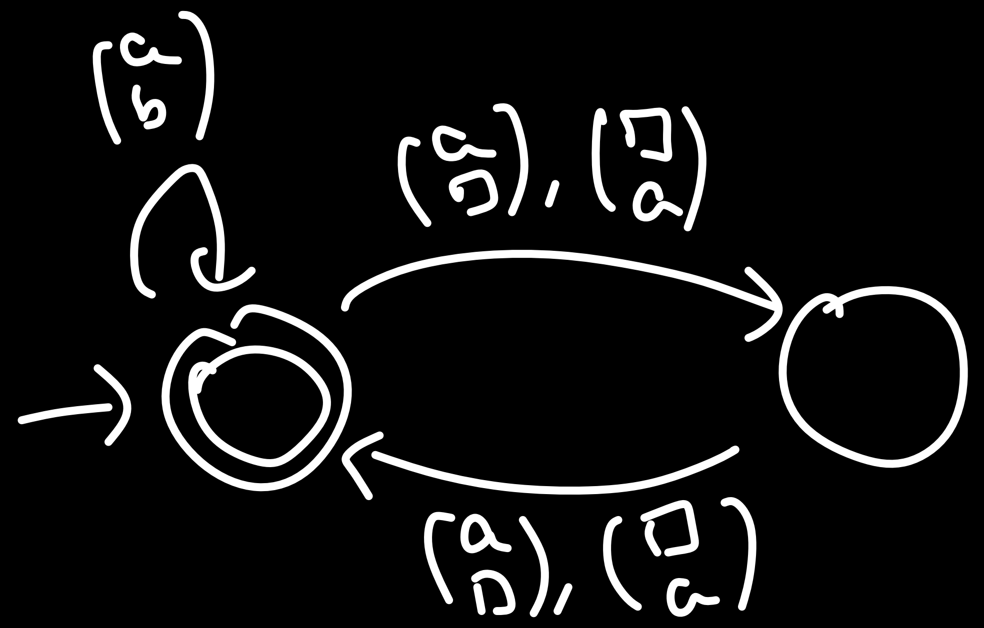


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# Relations vs. Languages

regular relations over $\Sigma$ $\subseteq$ regular languages over $\Sigma \times \Sigma \cup \Sigma \times \{\sqcup\} \cup \{\sqcup\} \times \Sigma$

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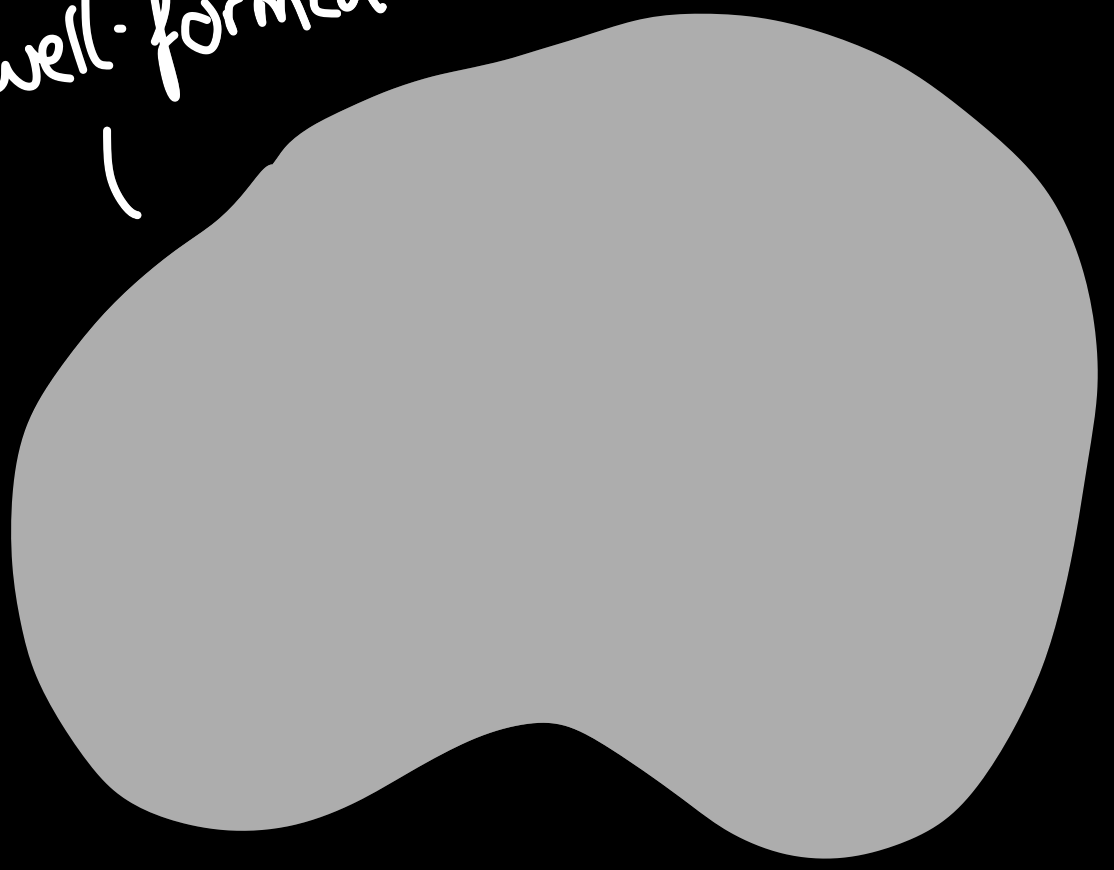
In $(\Sigma\times\Sigma \cup \Sigma\times\{\square\} \cup \{\square\}\times\Sigma)^*$ :

well-formed



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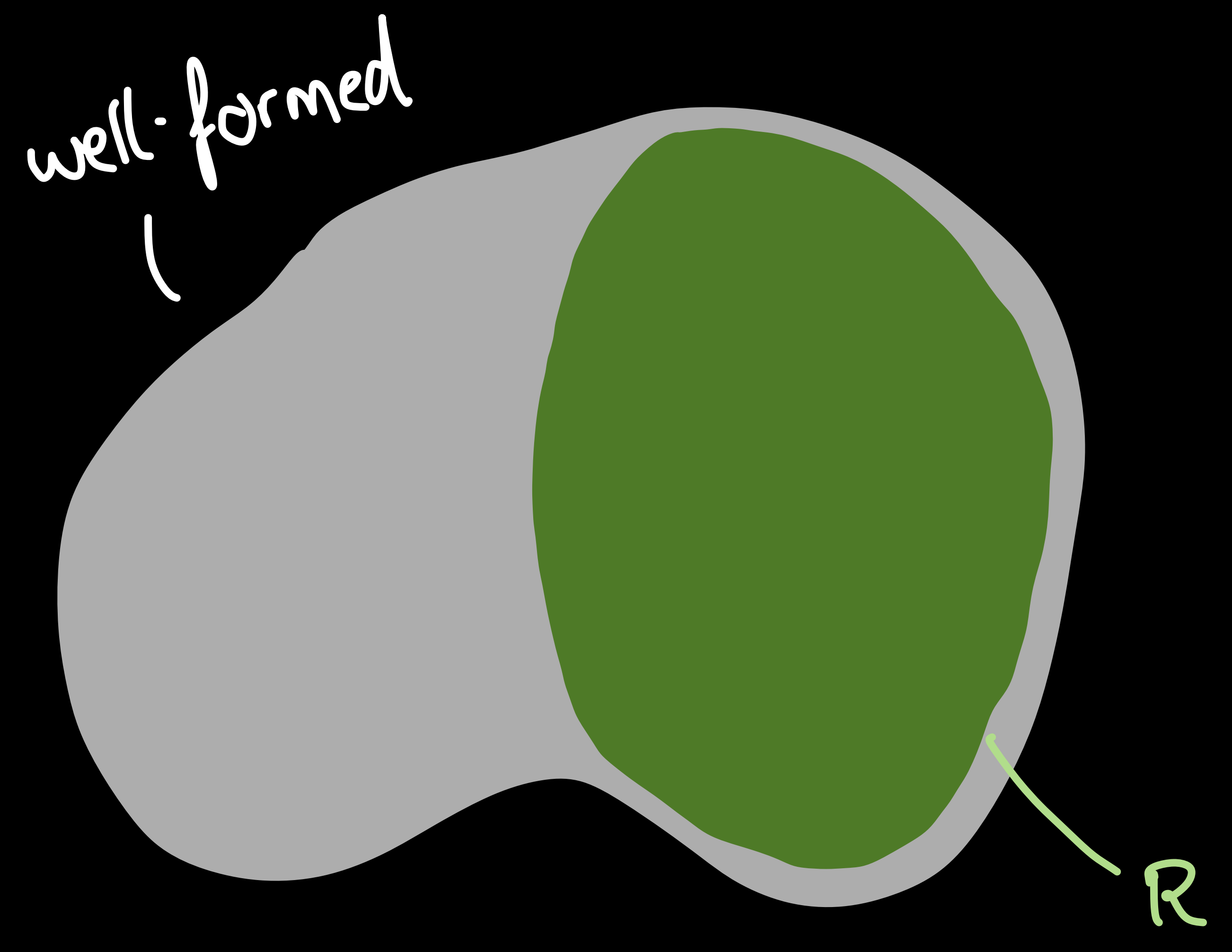

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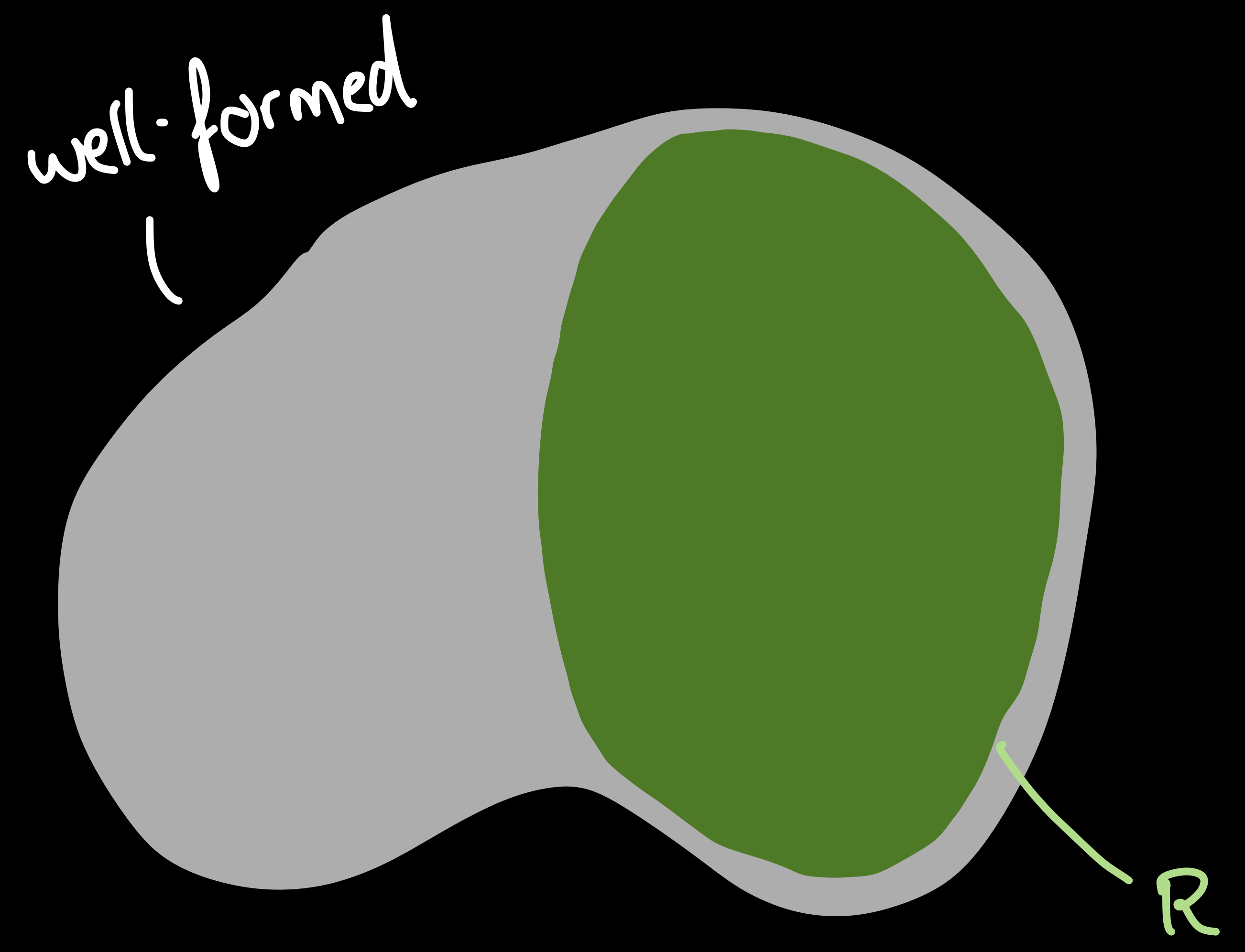


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same parity

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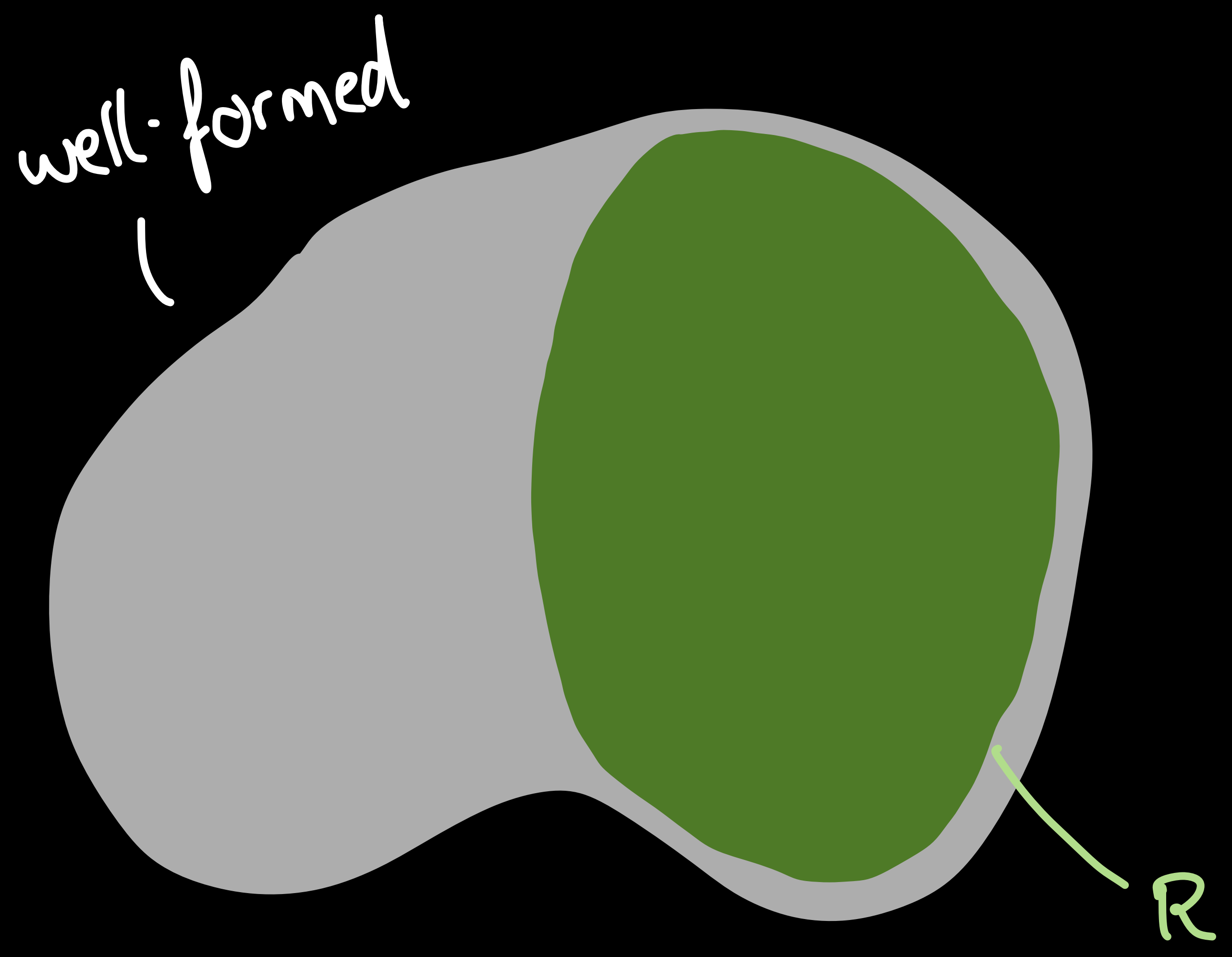


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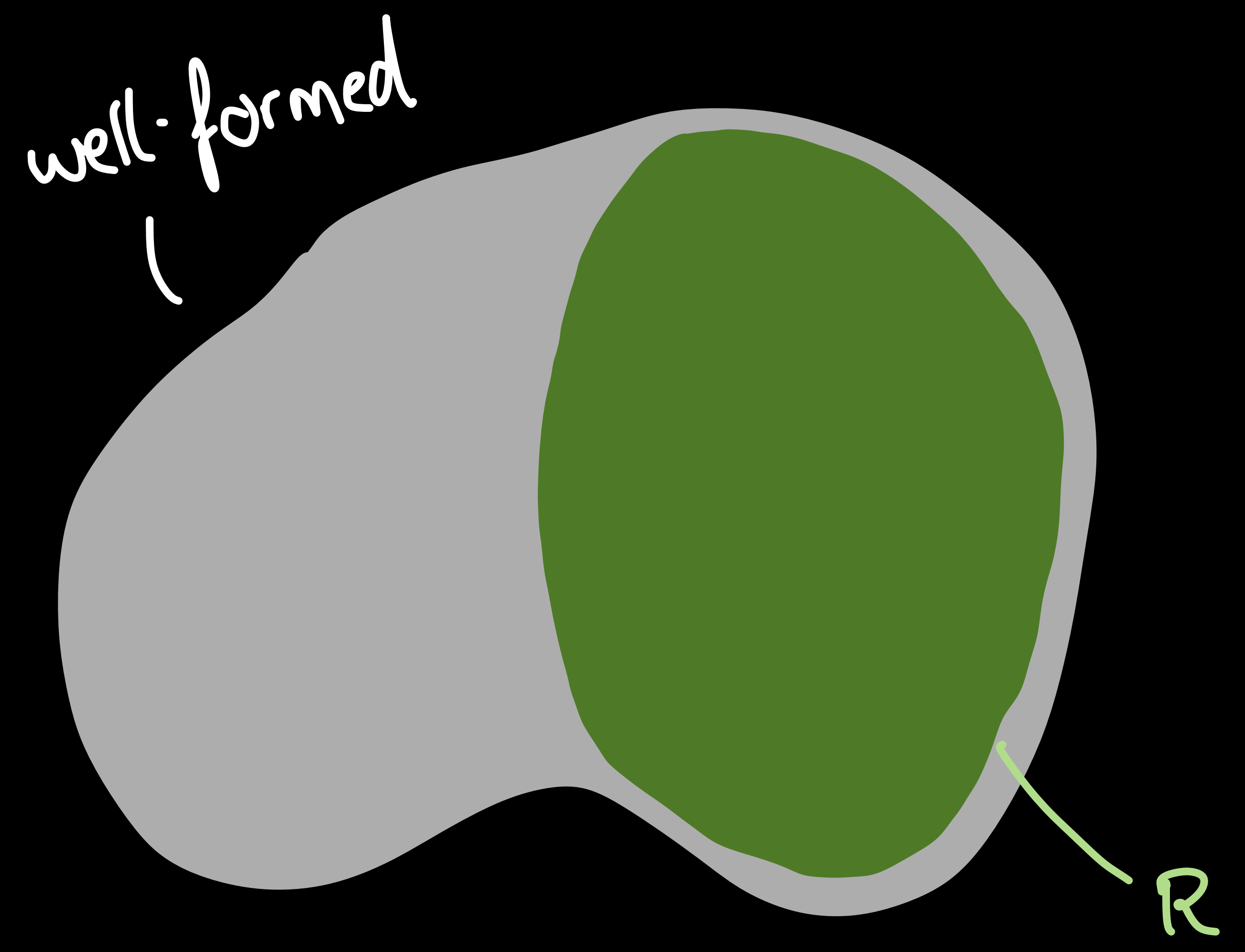


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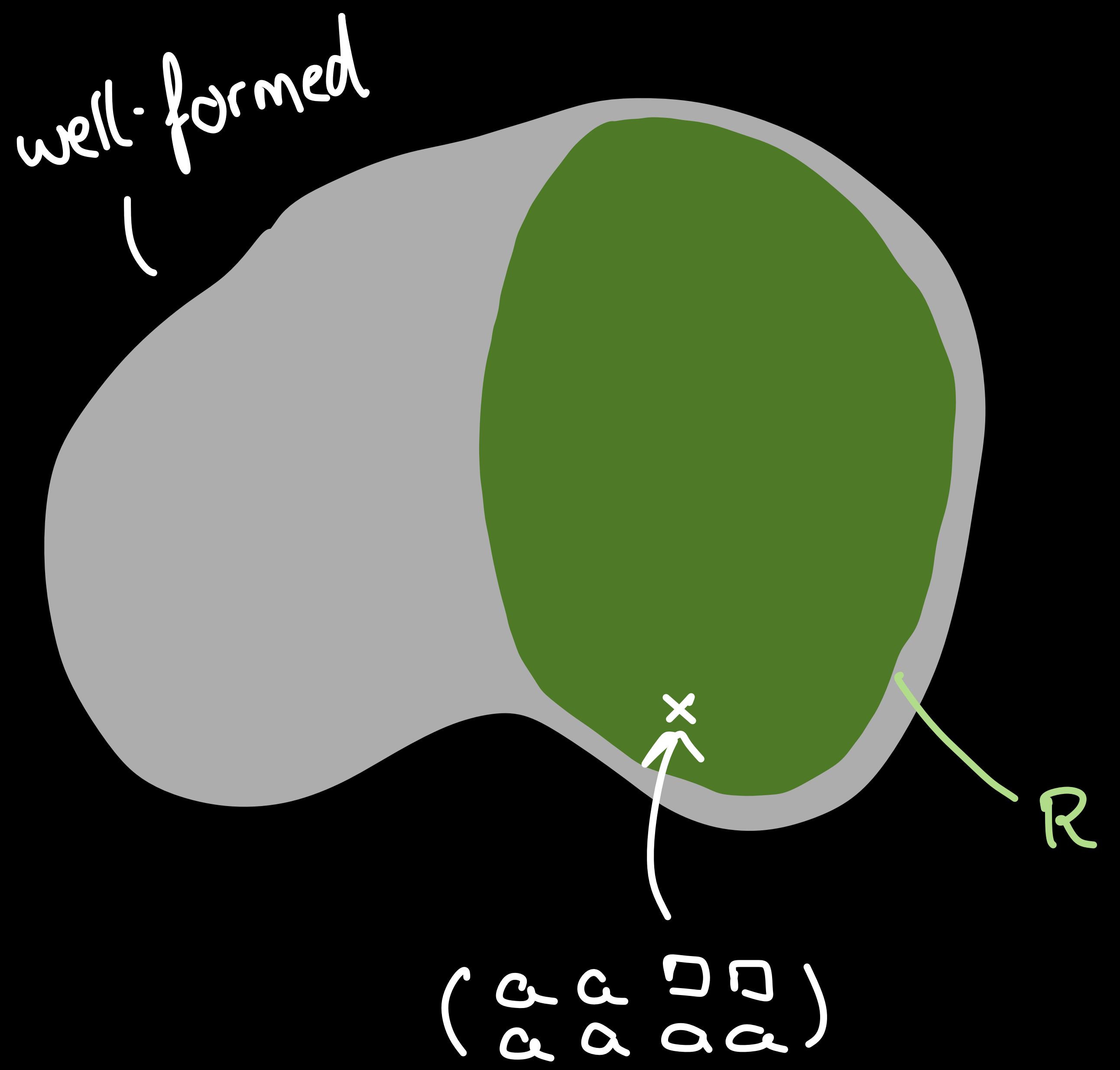


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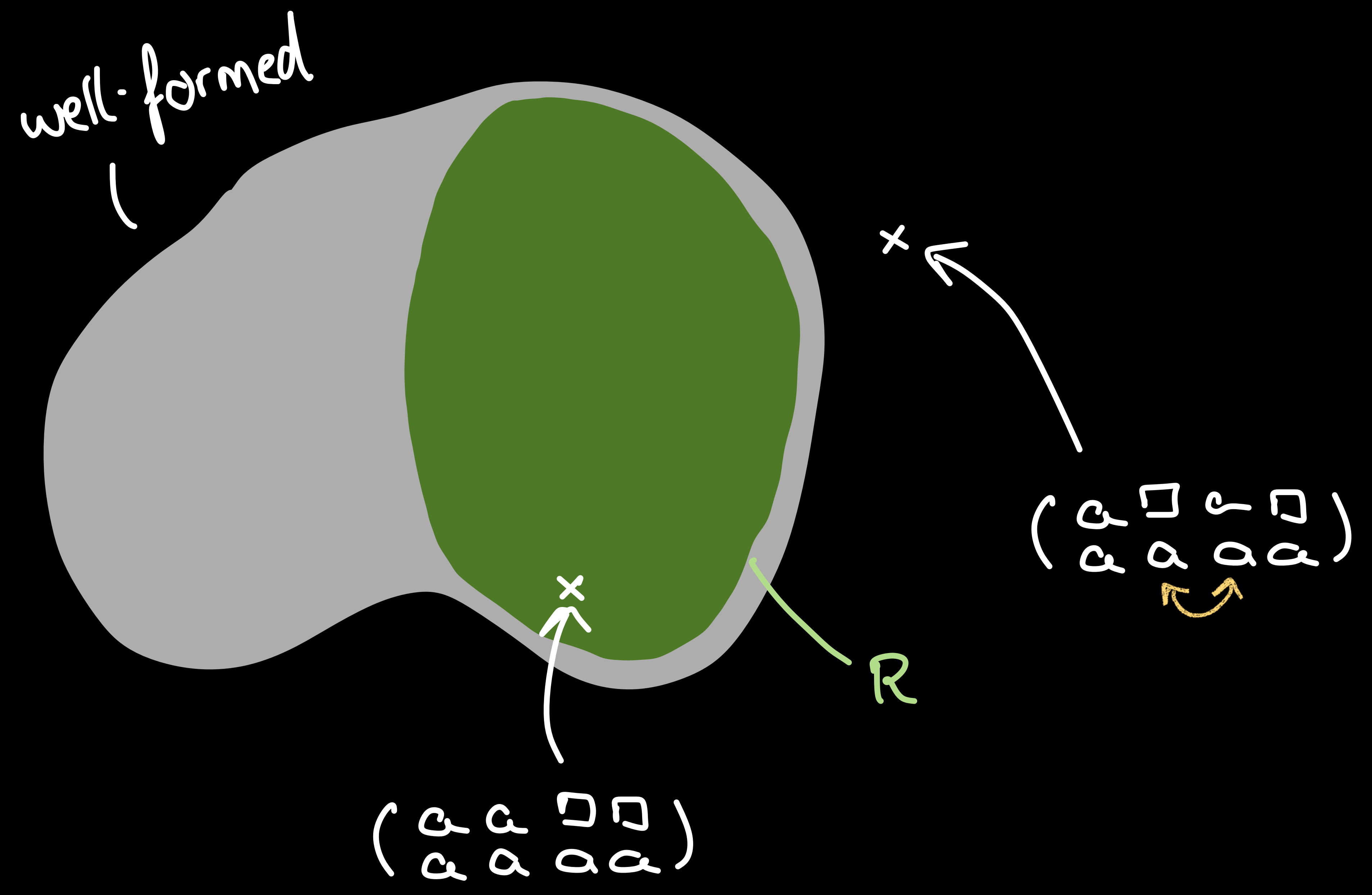


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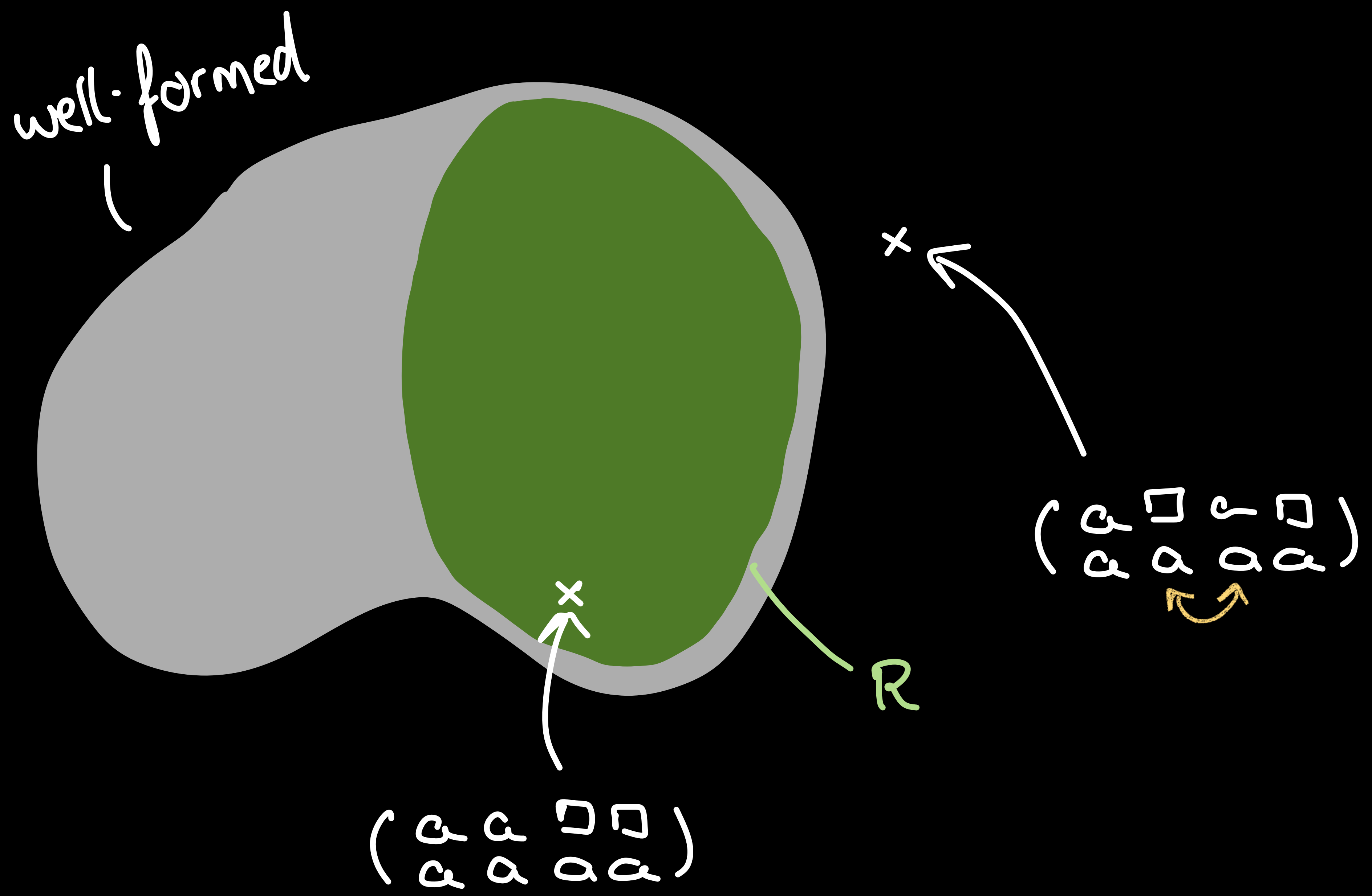


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<u>BUT</u>

$R = \left(\text{some commutative language}\right) \cap \left(\text{well-formed words}\right)$

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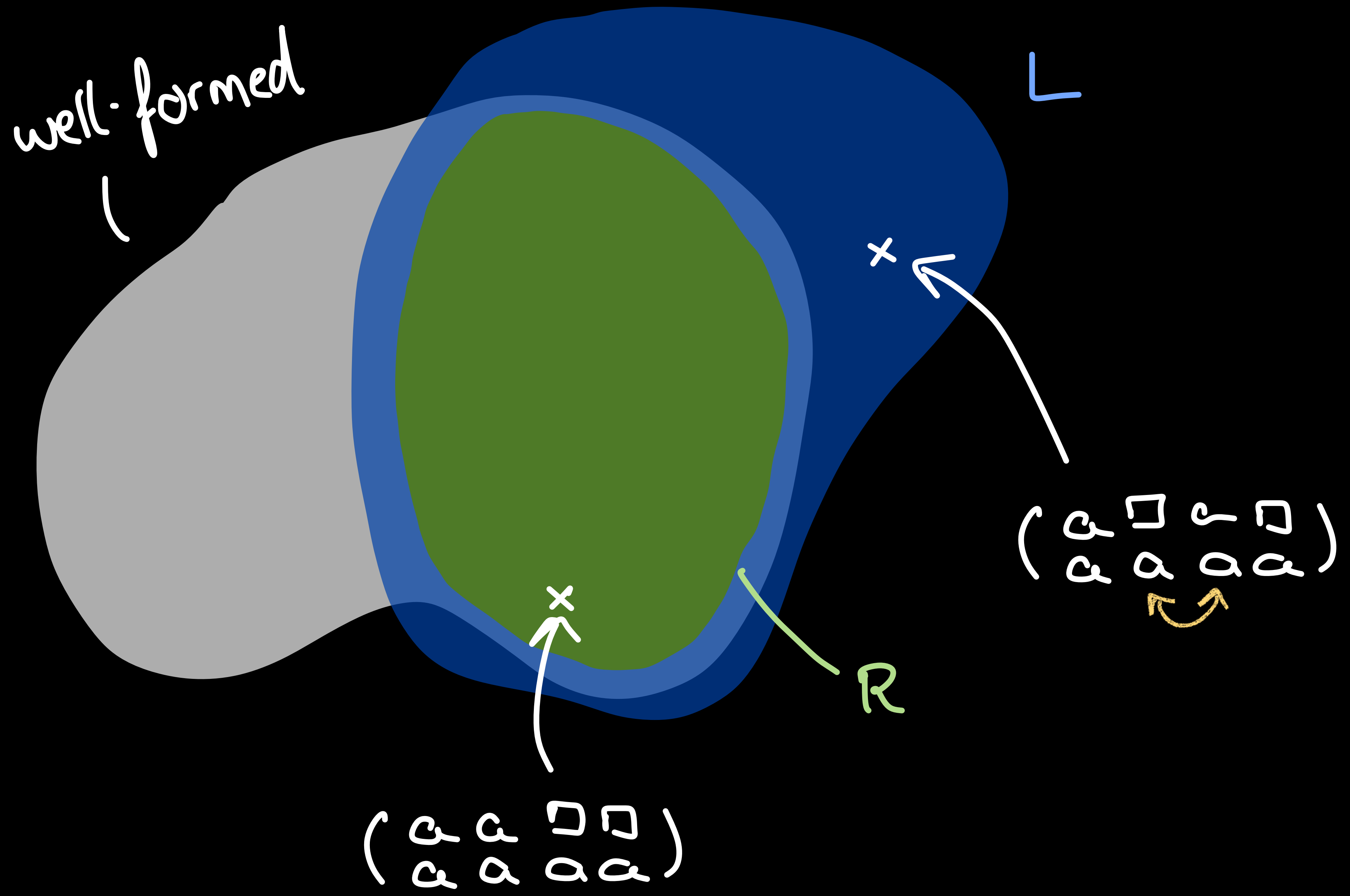


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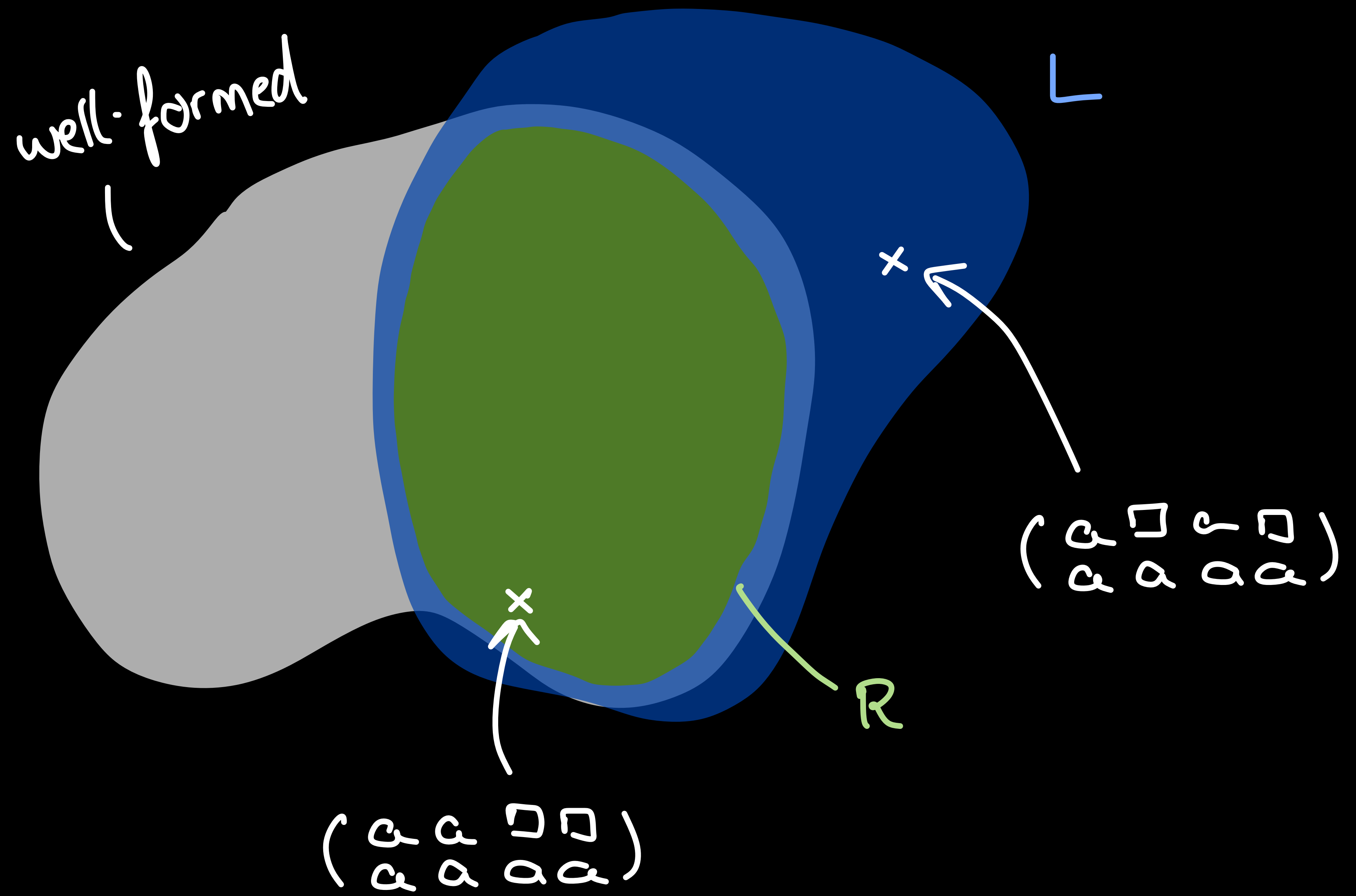

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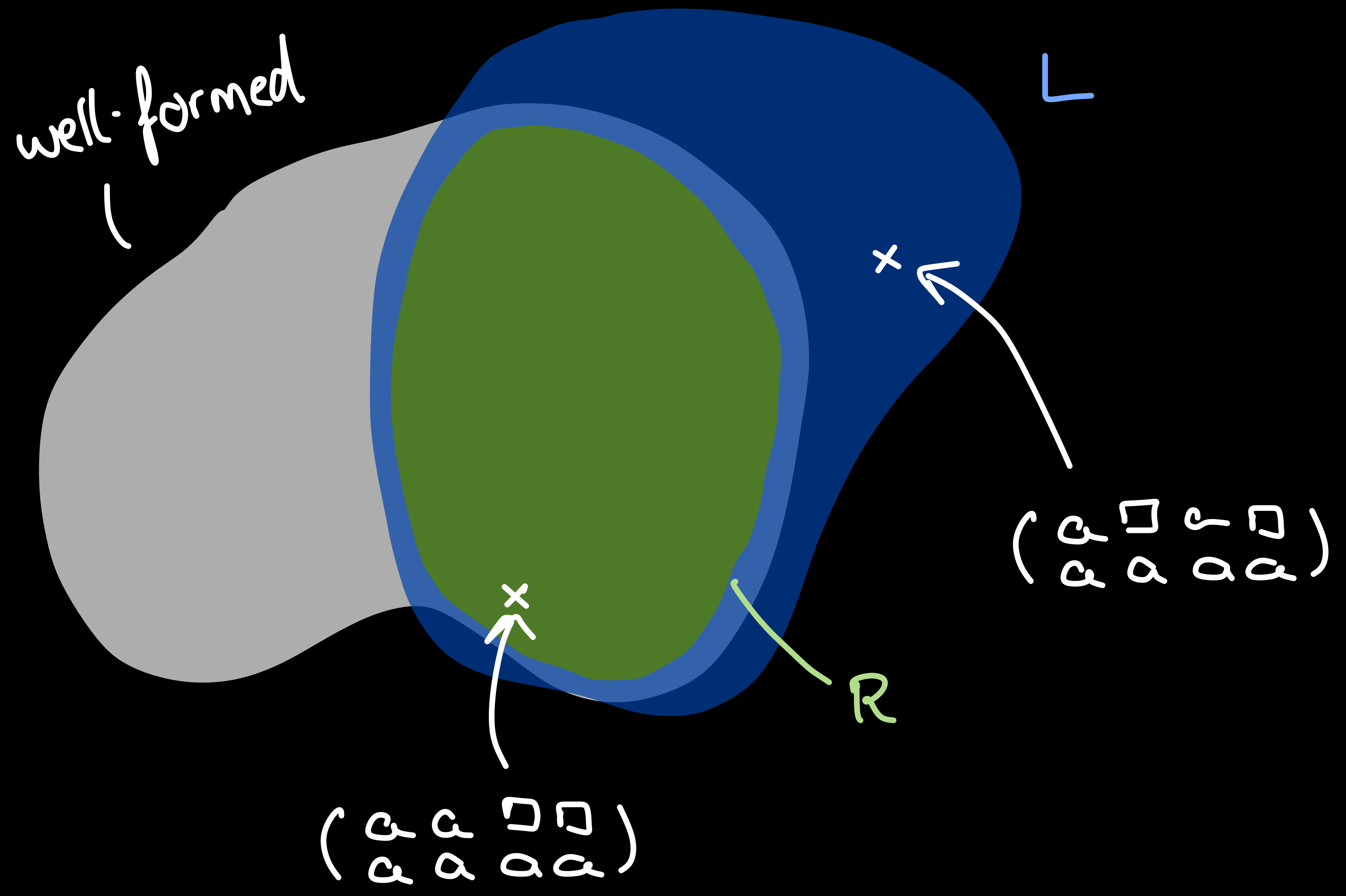
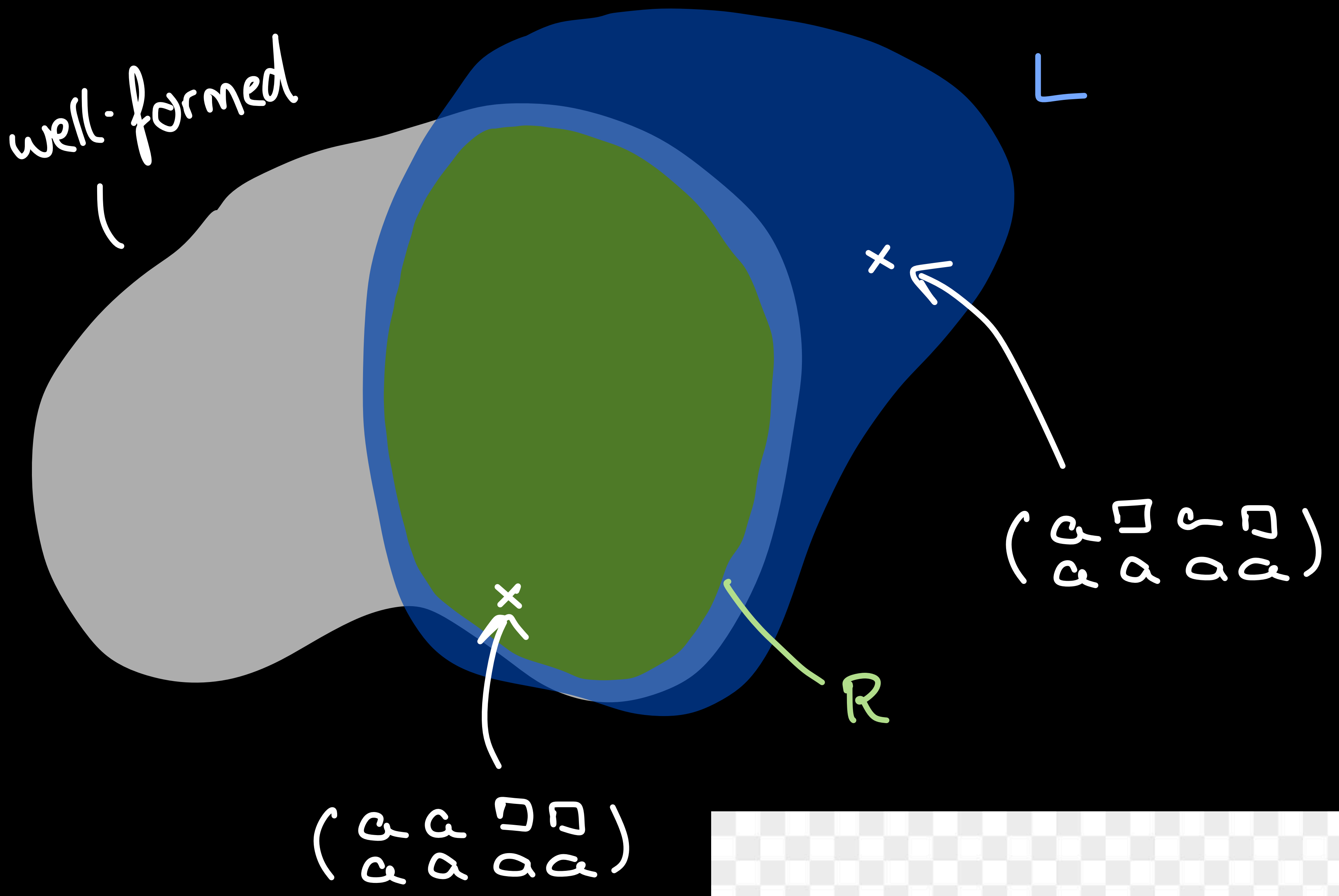


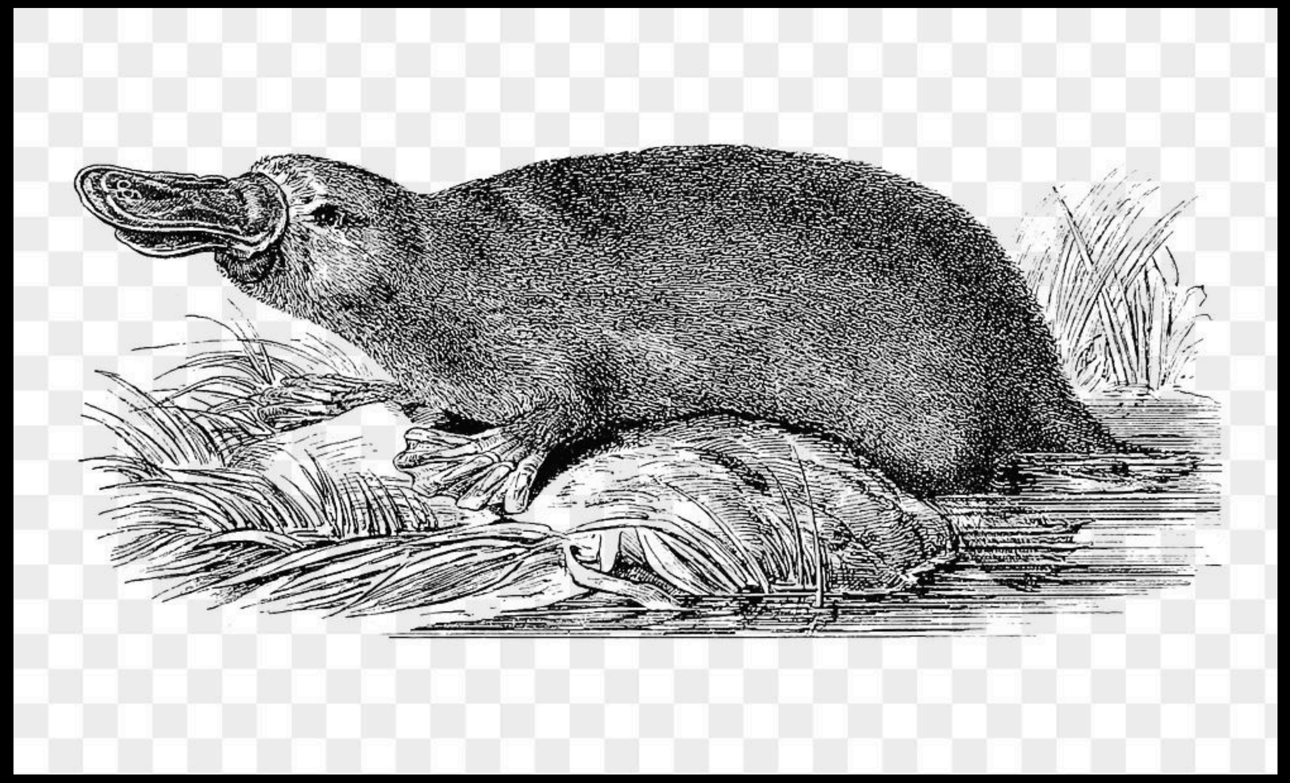

$\mathcal{V}$ : class of reg. languages

$Q^\circ$ : $\underbrace{\exists L \in \mathcal{V},\ R = L \cap (\text{well-formed words})\ ?}$

" "$\mathcal{V}$-relation" "

# Classes of Relations



$\mathcal{V}$: class of reg. languages

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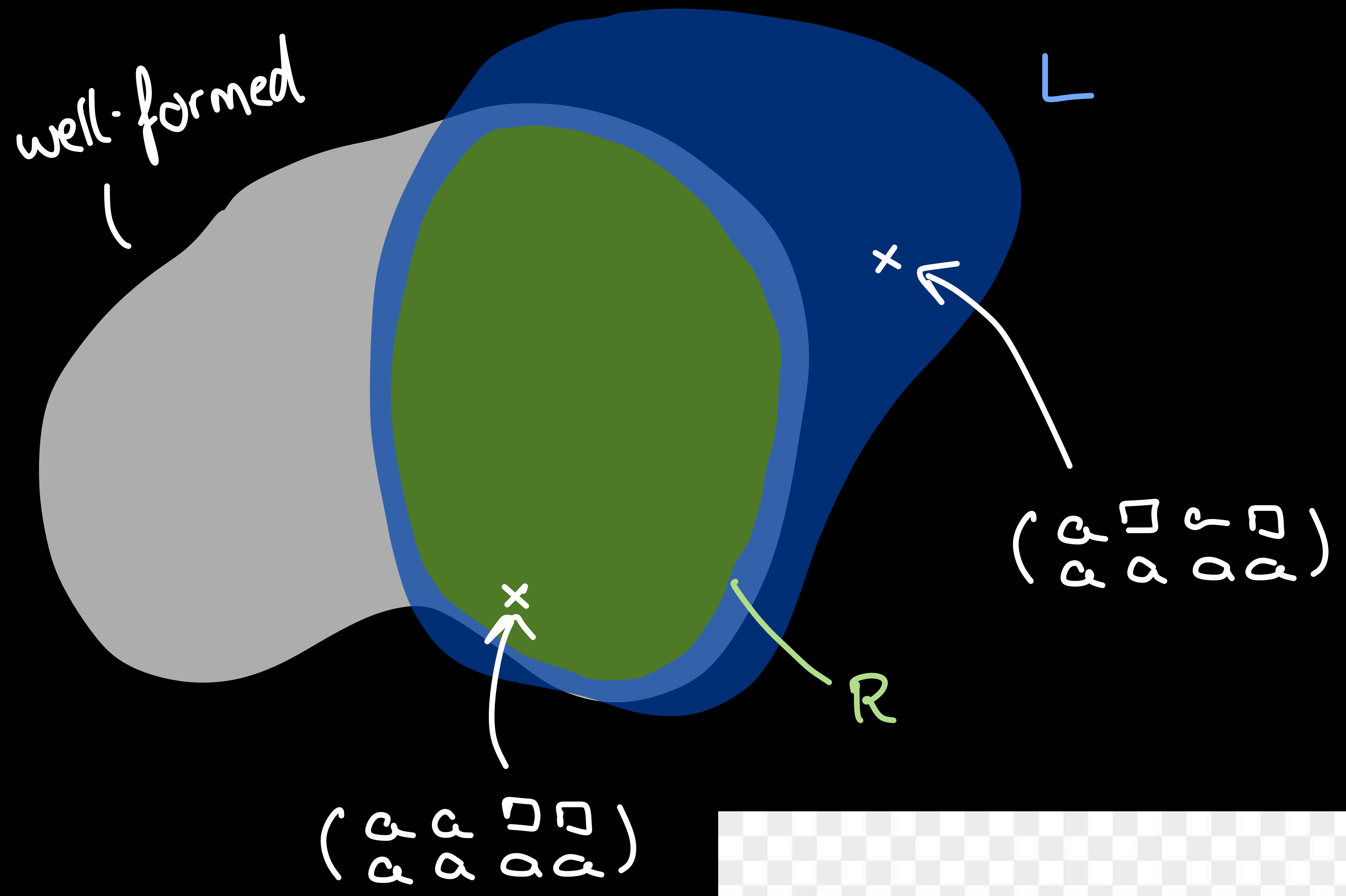


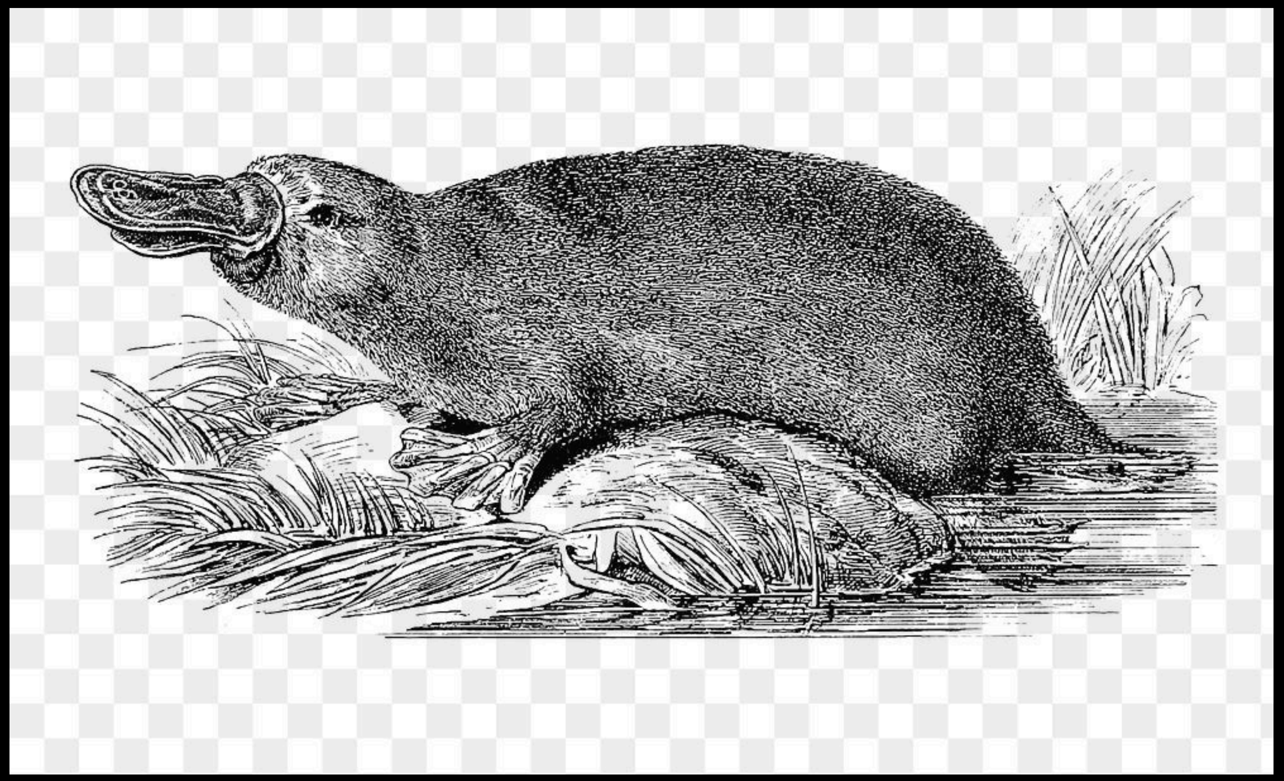

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Same parity is :

- not a commutative language ✗
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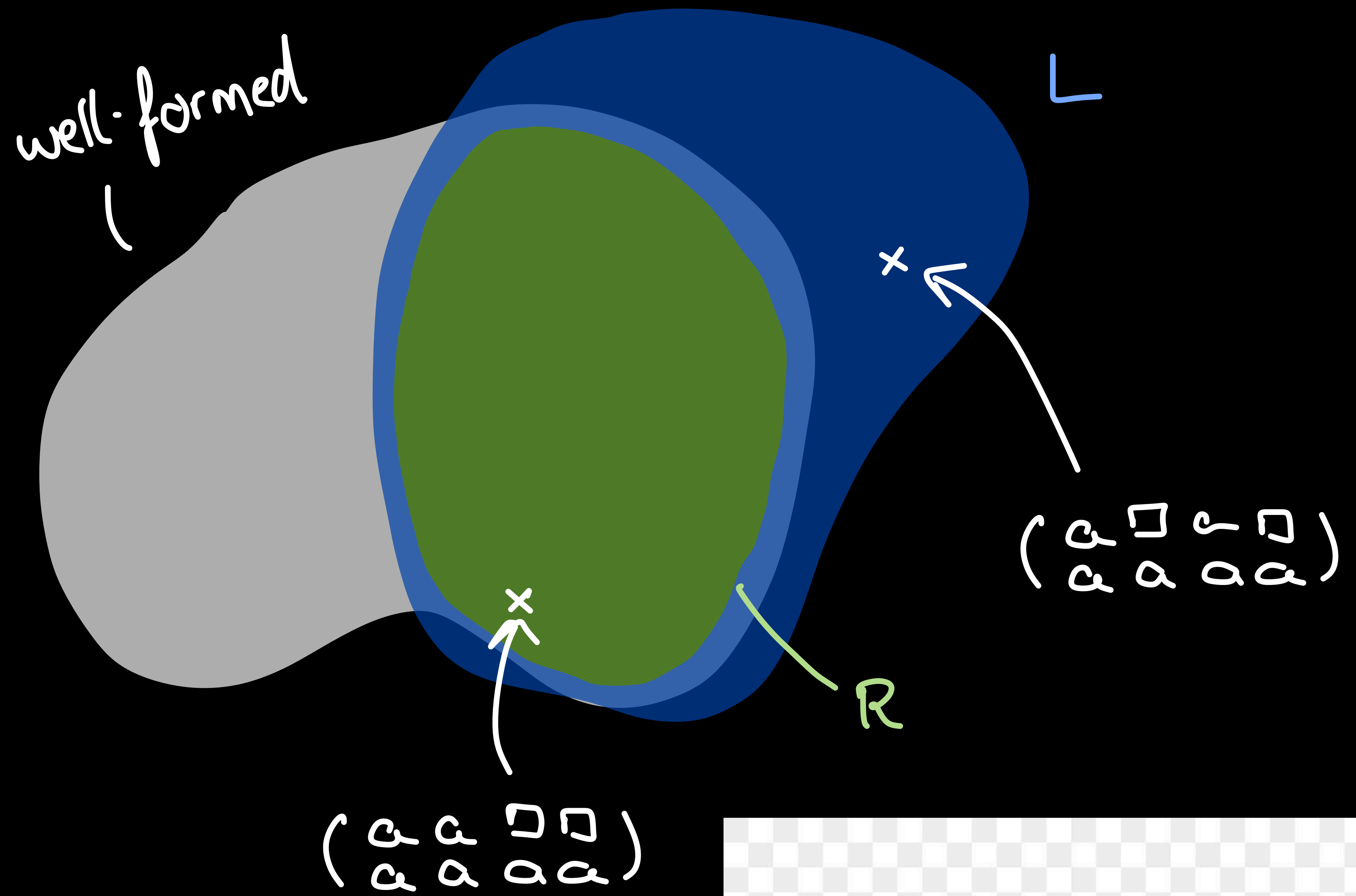
# Classes of Relations

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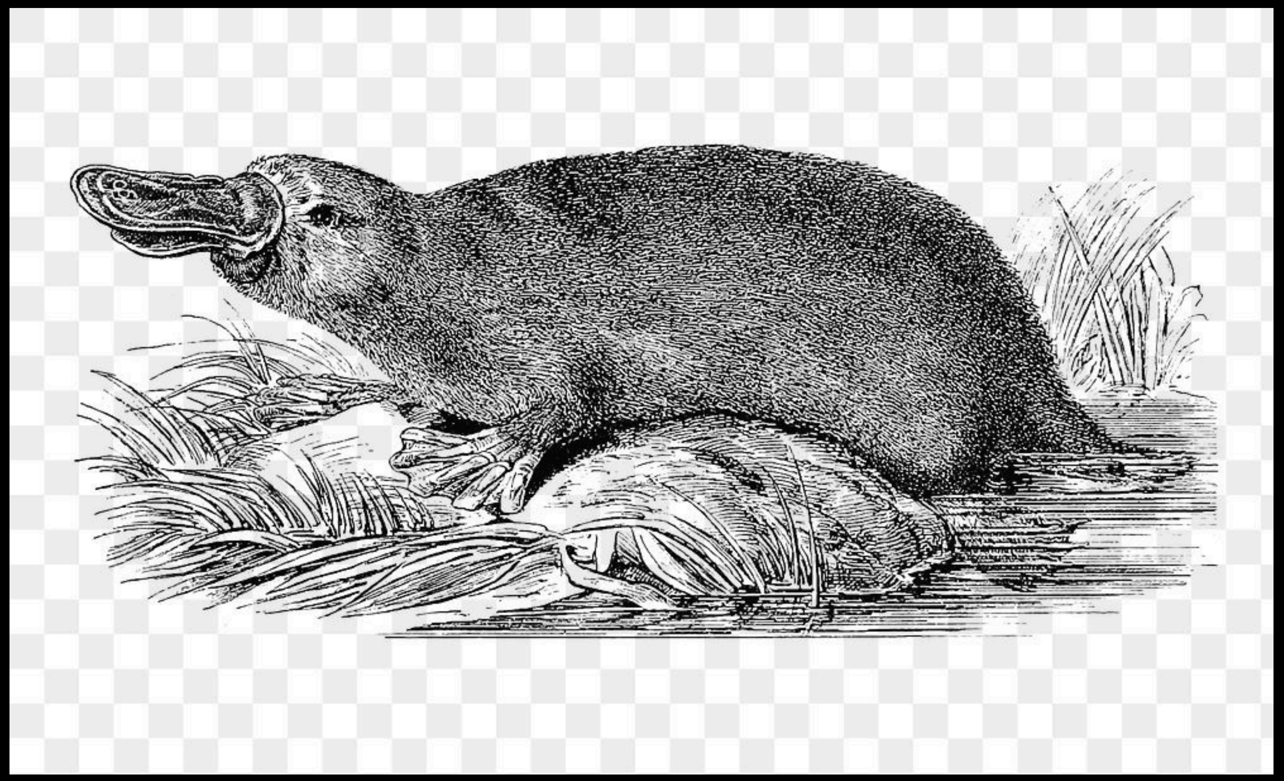

Same-parity is:

- not a commutative language ✗
- is a commutative relation ✓
- not a group language ✗
- is a group relation ✓

# Characterizing $\nabla$-relations

Classes of lang. $\nabla$

group languages

$\downarrow$

Classes of relations

group relations

# Characterizing $\mathcal{V}$-relations

Classes of lang. $\mathcal{V}$ $\longleftrightarrow$ Properties of semigroups $\in \mathbf{V}$

group languages

groups

$\downarrow$

Classes of relations

group relations

# Characterizing $\mathcal{V}$-relations

$$
\begin{array}{ccc}
\text{Classes of lang. } \mathcal{V} & \longleftrightarrow & \text{Properties of semigroups } \in \mathbf{V} \\
\big\downarrow & & \big\downarrow \\
\text{Classes of relations} & \longleftrightarrow & \text{Properties of synchronous algebras}
\end{array}
$$

group languages

groups

group relations

?

# Characterizing $\mathcal{V}$-relations

| | | |
|---|---|---|
| Classes of lang. $\mathcal{V}$ | $\longleftrightarrow$ | Properties of semigroups $\mathbb{V}$ |
| group languages | | groups |
| $\downarrow$ | | $\downarrow$ |
| Classes of relations | $\longleftrightarrow$ | Properties of synchronous algebras |
| group relations | | ? |

"Def": Synchronous algebras:

$\sim$ typed semigroups

$\binom{a}{a} \cdot \binom{a}{\square}$ ✓

$\begin{pmatrix} b & a \\ a & \square \end{pmatrix} \cdot \binom{a}{a}$ ✗

$x_\sigma \cdot y_\tau$ well-defined?

$\Updownarrow$

$(\sigma, \tau)$ compatible

# Characterizing $\mathcal{V}$-relations

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group languages — groups

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Classes of relations $\longleftrightarrow$ Properties of synchronous algebras

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**Fact** • Finitely many types

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$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

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- Monads à la Bojańczyk

# Characterizing $\mathcal{V}$-relations

Classes of lang. $\mathcal{V}$ $\longleftrightarrow$ Properties of semigroups $\mathbb{V}$

$\downarrow$ group languages $\qquad\qquad$ $\downarrow$ groups

$xy^! = x$,
$y^! x = x$ (profinite equalities)

Classes of relations $\longleftrightarrow$ Properties of synchronous algebras ?

group relations

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# Characterizing $\mathcal{V}$-relations

Classes of lang. $\mathcal{V}$ $\longleftrightarrow$ Properties of semigroups $\models \mathcal{V}$

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Classes of relations $\longleftrightarrow$ Properties of synchronous / chromatic algebras $\langle A, \equiv \rangle$

group relations $\qquad$ profinite equivalences $\qquad$ $xy^{!} \equiv x$, $y^{!}x \equiv x$

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# Characterizing $\mathcal{V}$-relations

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group languages

groups

$xy^! = x$,
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(profinite equalities)

$\downarrow$ $\qquad\qquad$ $\downarrow$

Classes of relations $\longleftrightarrow$ Properties of synchronous chromatic algebras

$\langle A, \equiv \rangle$

group relations

profinite equivalences

$xy^! \equiv x$,
$y^! x \equiv x$

**Thm** This construction works for many $\mathcal{V}$.

**Conj** Works for all $\mathcal{V}$?

**"Def"**: Synchronous algebras:

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