Universal algorithms for parity games and nested fixpoints

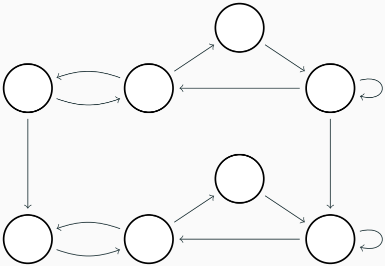
ANR DELTA meeting

Marcin Jurdziński[1], Rémi Morvan[2], K. S. Thejaswini[1]

June 28, 2021, in Paris!

[1]University of Warwick
[2]École normale supérieure Paris-Saclay

Parity games



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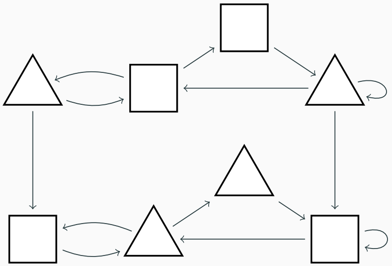

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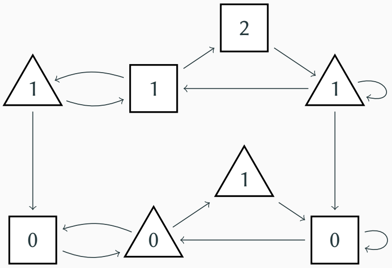

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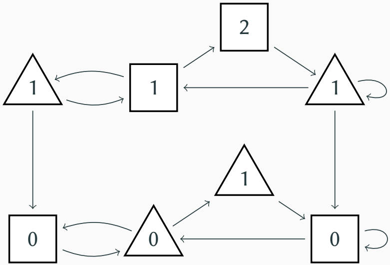

- Play: $(v_i)_{i \in \mathbb{N}}$ s.t. $\forall i,\ (v_i, v_{i+1}) \in E$.

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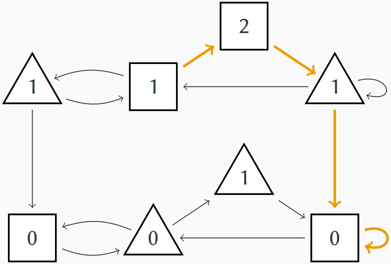

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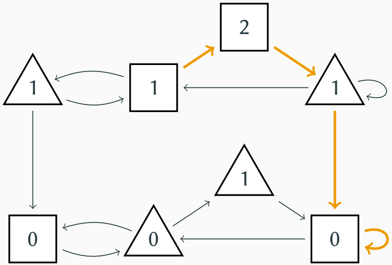

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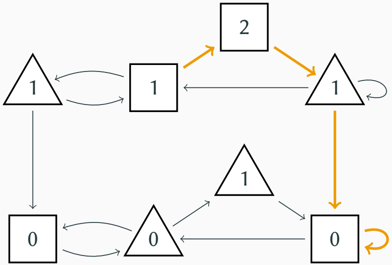

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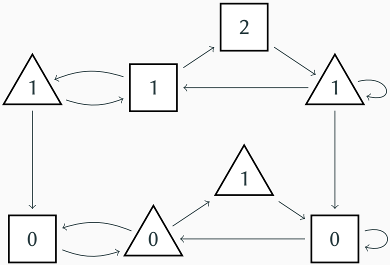

- Memoryless strategy for Odd: $\sigma : V_{\text{Odd}} \rightharpoonup V$ s.t. $(v, \sigma(v)) \in E$ for every vertex v.

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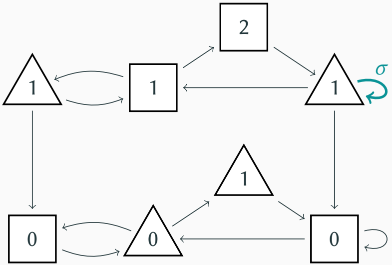

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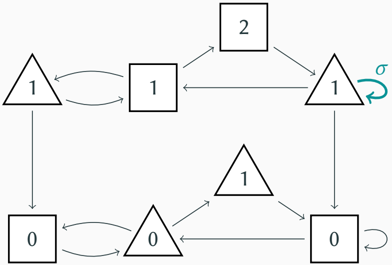

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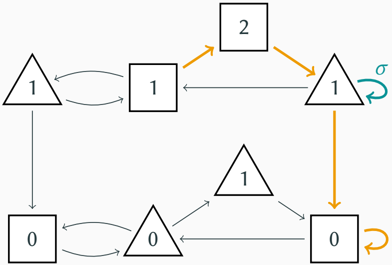

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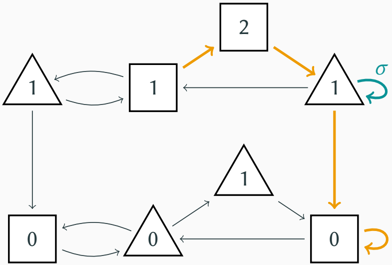

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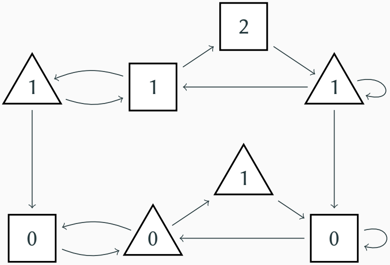

Theorem: From every vertex $v_0 \in V$, one of the two players has a memoryless strategy σ such that every play consistent with σ and starting at v_0 is winning for her.

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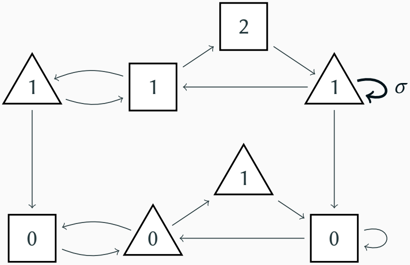

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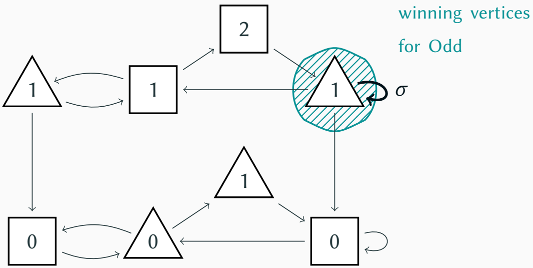

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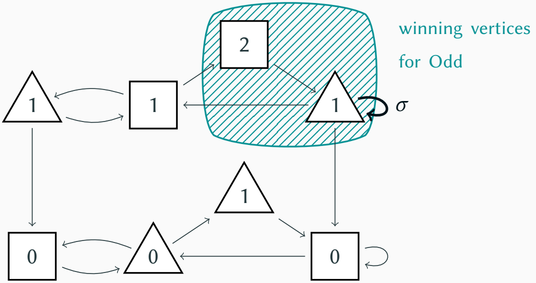

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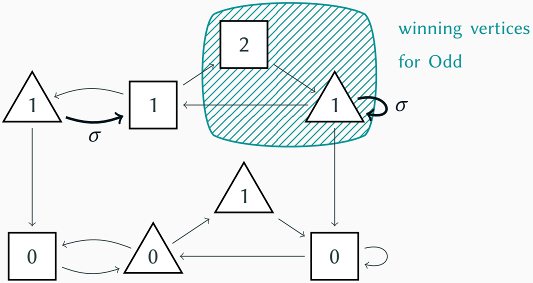

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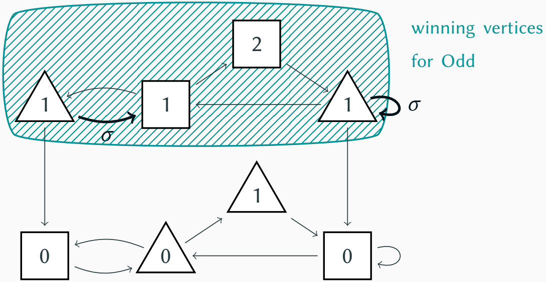

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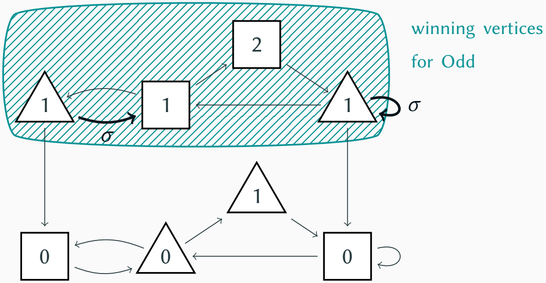

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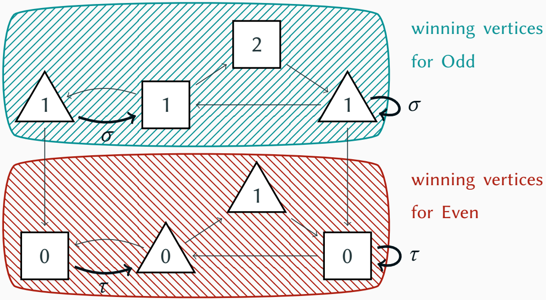

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Solving parity games

SOLVING PARITY GAMES:

Inputs: $\mathcal{G}$: parity game,

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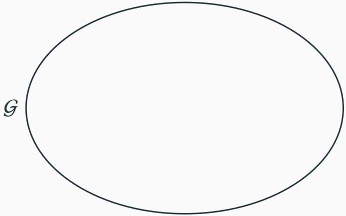
- Best known upper bound: quasipolynomial time $O(n^{\log(d)})$ ['17 Calude-Jain-Khoussainov-Li-Stephan]

A recursive algorithm?



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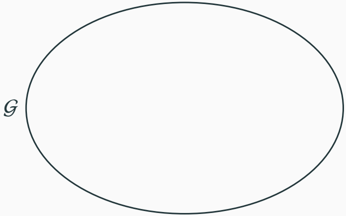

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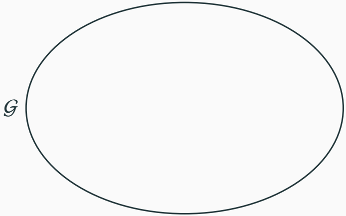

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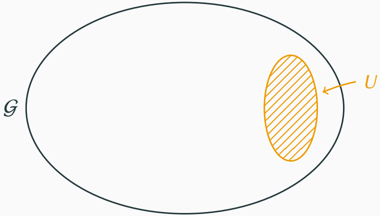

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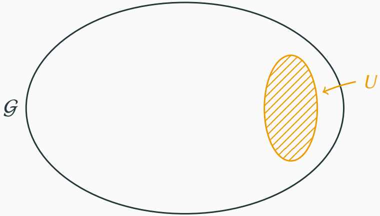

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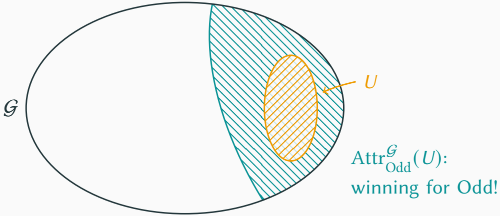

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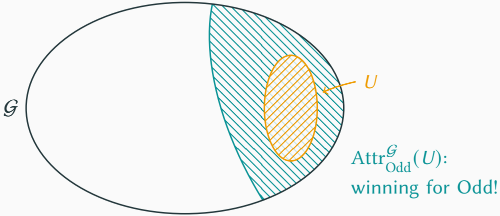

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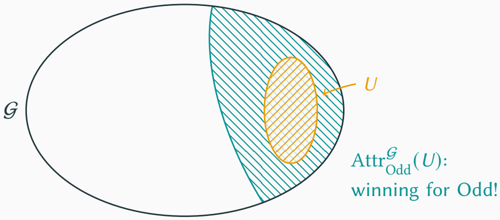

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  - none ← easy to identify!
  - at least one, but finitely many
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How to compute the set of v s.t. Odd can win from v without ever seeing a vertex of priority d? It is the set of winning vertices for Odd in the game

$$\mathcal{G} \setminus \mathrm{Attr}^{\mathcal{G}}_{\mathrm{Even}}(\pi^{-1}[d]).$$

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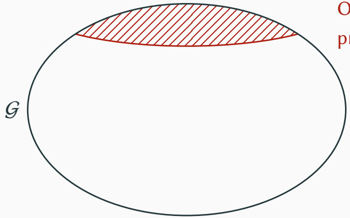

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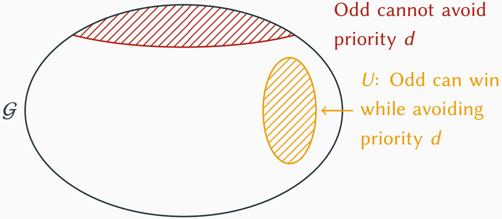

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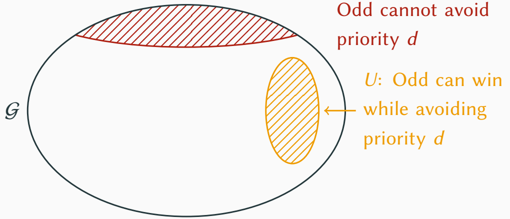

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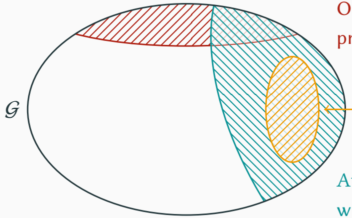

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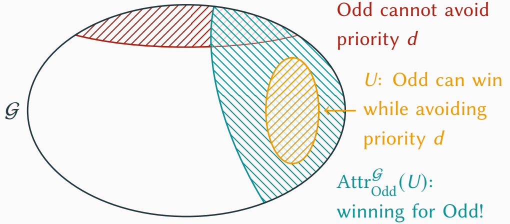

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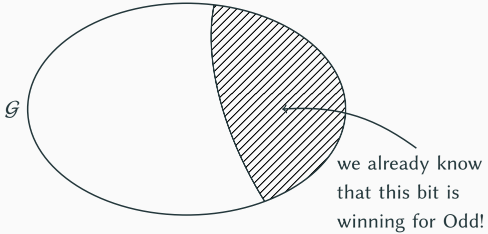

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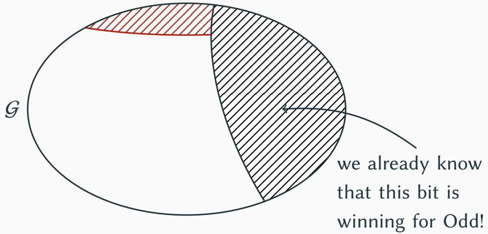

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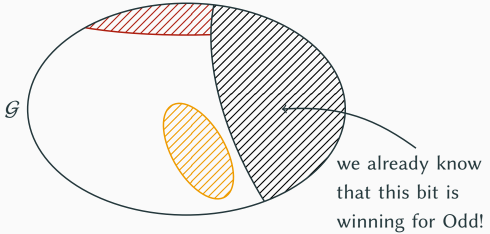

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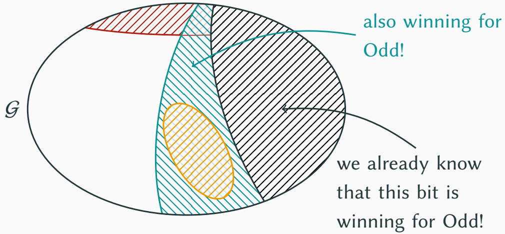

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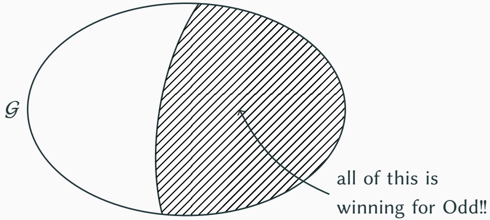

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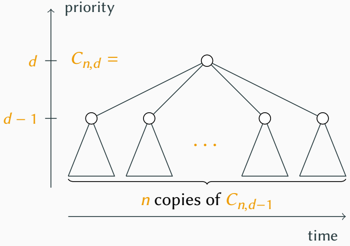

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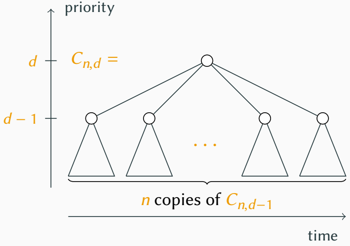

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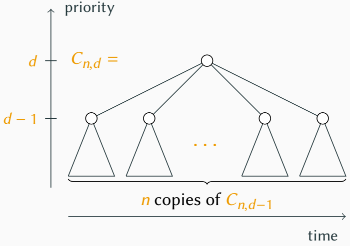

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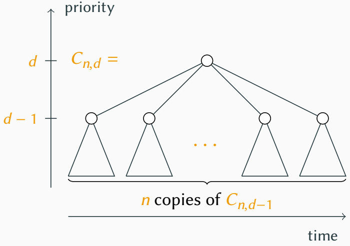

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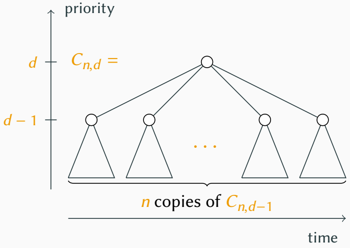

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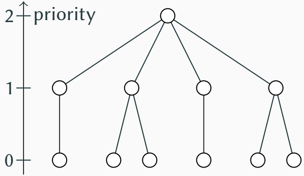

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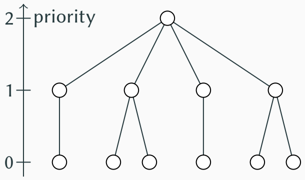

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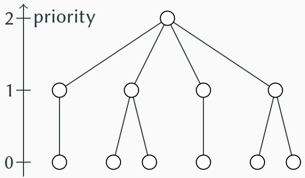

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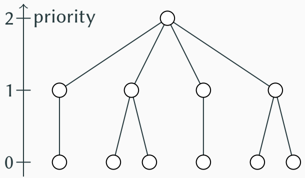

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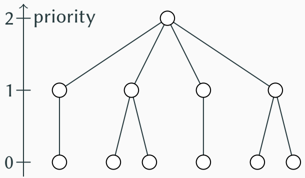

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   Iterate k times, where k is the number of children of the root of $\mathcal{T}$.

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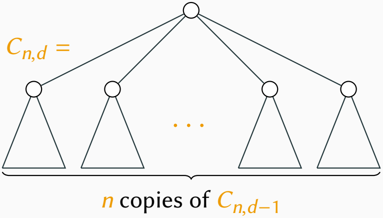

Parys & Lehtinen-Schewe-Wojtczak

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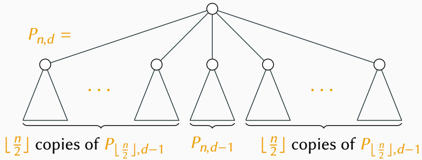

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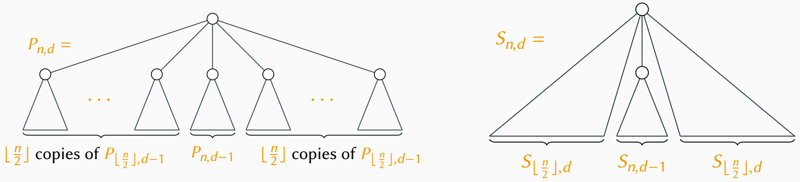

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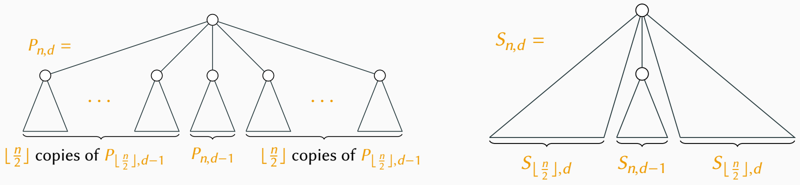

n: number of vertices & d: top priority

Universal algorithm: correctness & complexity

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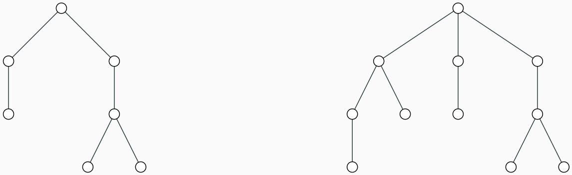
Ordered trees



- Ordered trees: partially ordered by the “embedding” relation.

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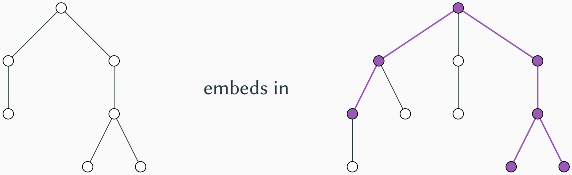

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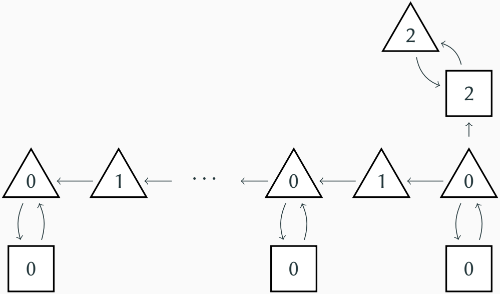
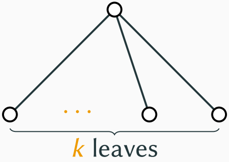
Attractor decomposition





An attractor decomposition $\mathrm{Win}_{\mathrm{Even}}(\mathcal{G})$ of $\mathcal{G}$ is a partition of the winning set describing the structure of Even's winning set. [Daviaud-Jurdziński-Lehtinen, '18]

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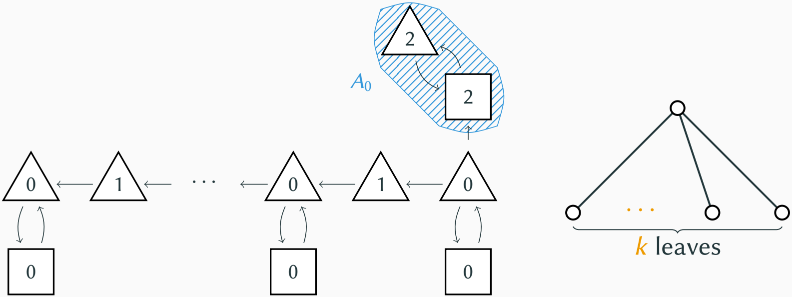

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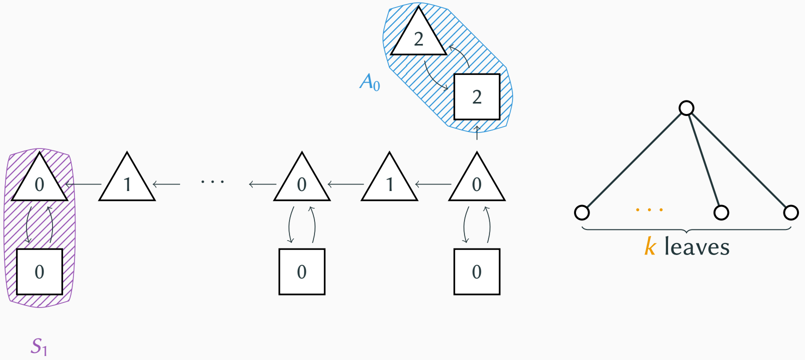

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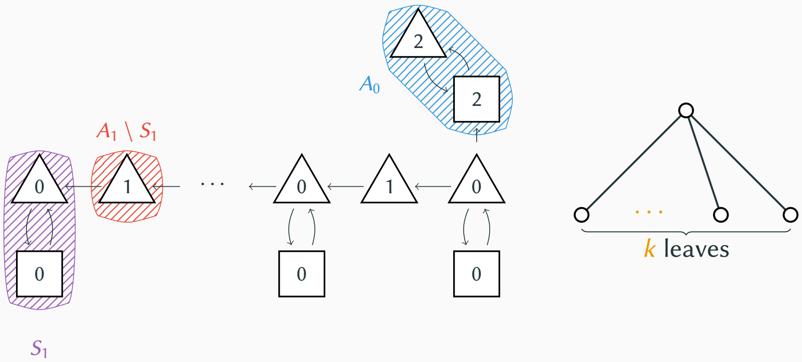

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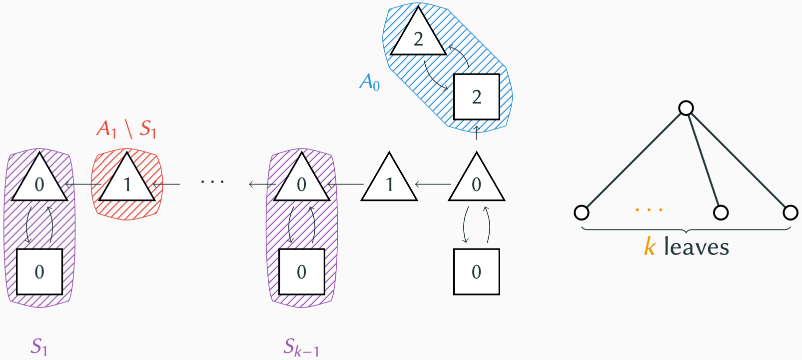

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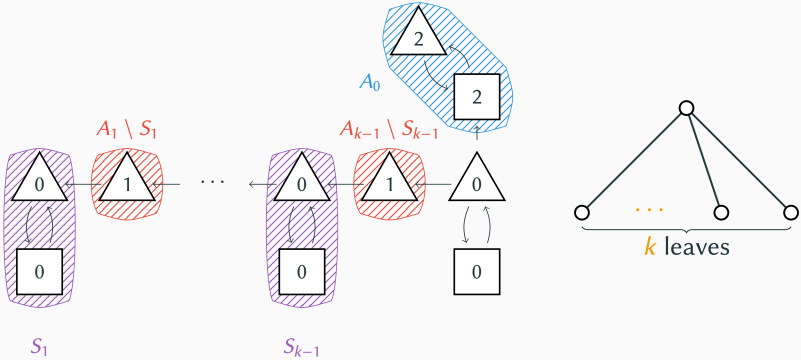

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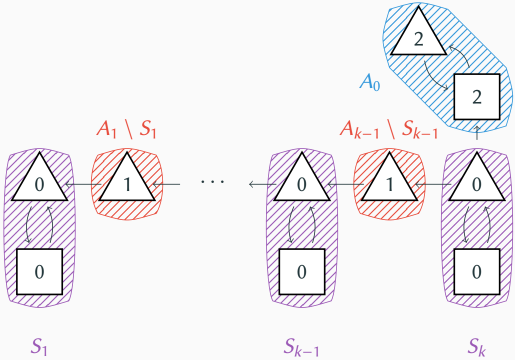

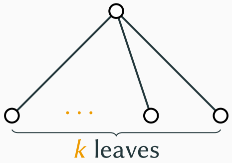

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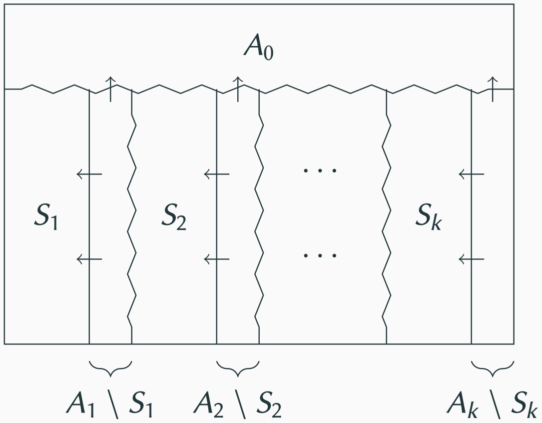
Attractor decomposition (bis)



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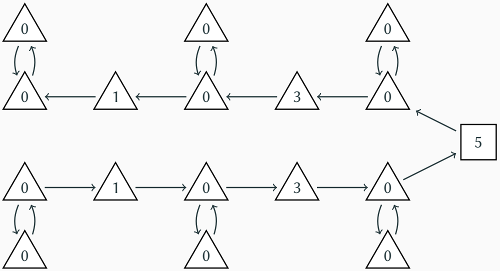

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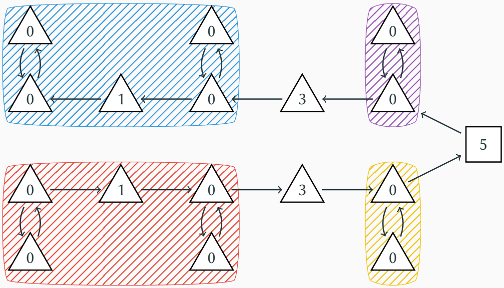

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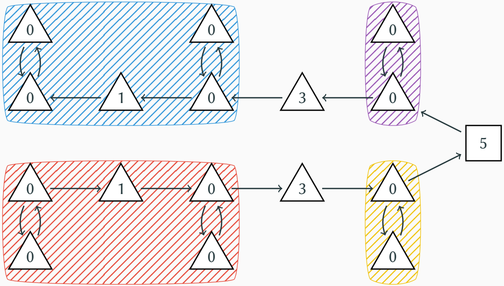

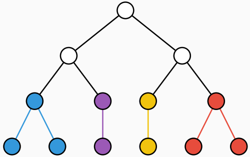

Embeddable decomposition theorem

Theorem: If D is subset of the winning set W for Even, if Odd can force the play to stay in D, for every attractor decomposition tree $\mathcal{T}_W$ of W, there exists an attractor decomposition tree $\mathcal{T}_D$ of D such that: $\mathcal{T}_D$ embeds in $\mathcal{T}_W$.

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Attractor decomposition trees describe the shape of the structure of a winning region.

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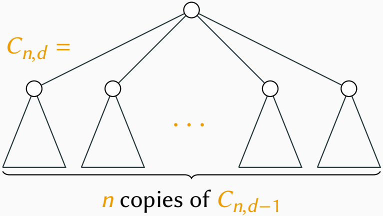

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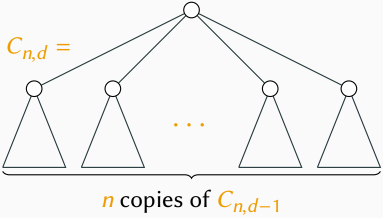

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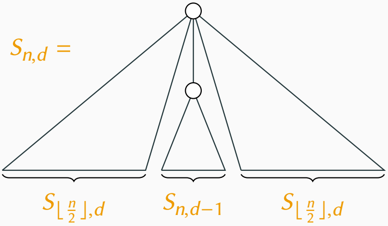

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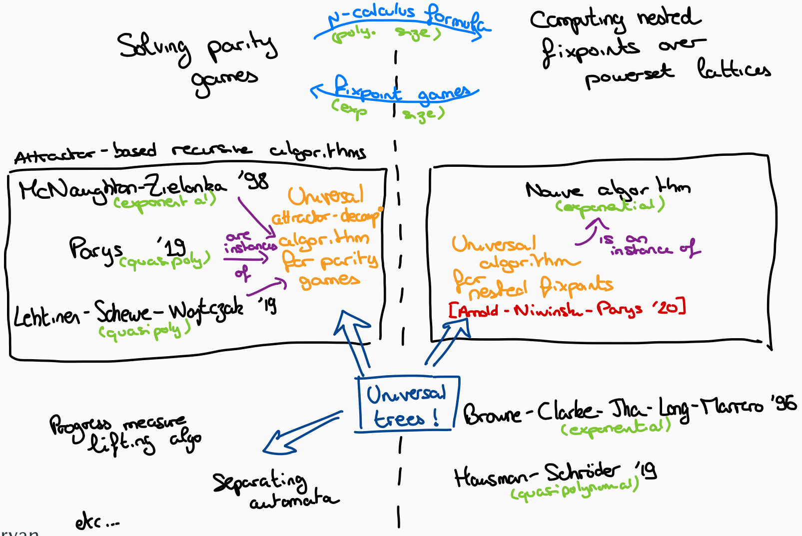
- This applies to McNaughton-Zielonka '98, to Parys '19 and to Lehtinen-Schewe- Wojtczak '19.

Conclusion


Solving parity games
μ-calculus formula
(poly. size)
Fixpoint games
(exp size)
Computing nested fixpoints over powerset lattices
Attractor-based recursive algorithms
McNaughton-Zielonka '98
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[Arnold-Niwinski-Parys '20]
Universal trees!
Progress measure lifting algo
Separating automata
etc...
Browne-Clarke-Jha-Long-Marrero '96
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Hausman-Schröder '19
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