Separation over infinite words

ANR DELTA, online

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based on works with
Thomas Colcombet & Sam van Gool.

4 January, 2022

First-order logic (FO)

Let $w \in A^*$ where $A = \{a, b, c, \ldots\}$.

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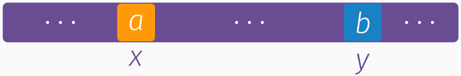

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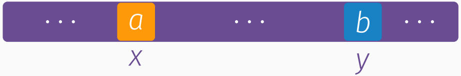

i.e. $w \in A^* a A^* b A^*$.

FO-definability

FO-DEFINABILITY:
Input: Morphism $f : A^* \to M$
Question: Is f **FO-definable**?

$$f(u) := \begin{cases} m_1 & \text{if } u \models \varphi_1 \\ m_2 & \text{if } u \models \varphi_2 \\ \vdots & \vdots \\ m_n & \text{if } u \models \varphi_n \end{cases}$$

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Theorem [Schützenberger '65 & McNaughton-Papert '71]:
A morphism $f : A^* \to M$ is FO-definable IFF $\operatorname{im} f$ is aperiodic.

Corollary: FO-DEFINABILITY is decidable.

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Theorem [Schützenberger '65 & McNaughton-Papert '71]:
A morphism $f : A^* \to M$ is FO-definable IFF $\operatorname{im} f$ is aperiodic.

every group in $\operatorname{im} f$ is trivial

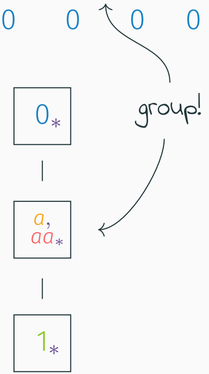
Corollary: FO-DEFINABILITY is decidable.

Example!

$$f : \{a, b\}^* \to M$$

$$u \mapsto \begin{cases} 1 & \text{if } u = \varepsilon \\ a & \text{if } u \in a(aa)^*, \\ aa & \text{if } u \in (aa)^+, \\ 0 & \text{if } u \text{ contains a 'b'} \end{cases}$$

$\cdot$	1	a	aa	0
1	1	a	aa	0
a	a	aa	a	0
aa	aa	a	aa	0
0	0	0	0	0



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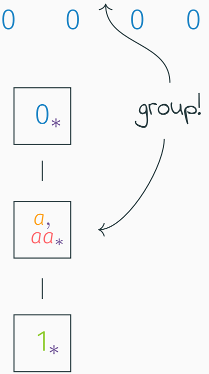
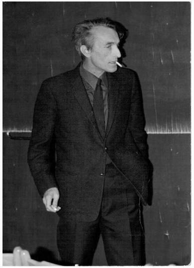


f is not FO-definable...
but still carries "FO-describable information"

Qualitative vs. quantitative

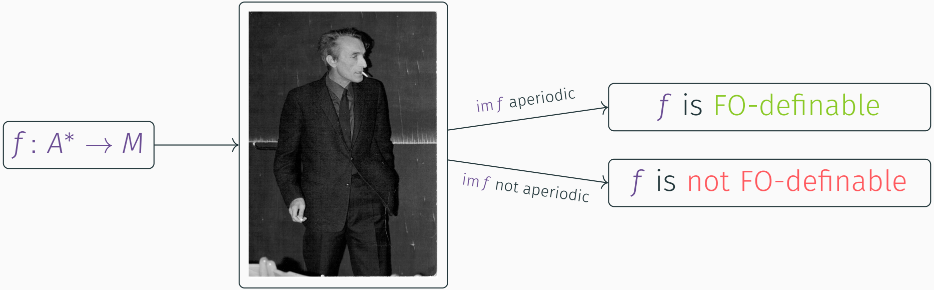
$f : A^* \to M$



$\operatorname{im} f$ aperiodic → f is FO-definable

$\operatorname{im} f$ not aperiodic → f is not FO-definable

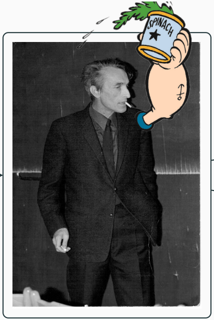
Qualitative vs. quantitative



Can we make a quantitative version of Schützenberger-McNaughton-Papert?

Qualitative vs. quantitative

$f : A^* \to M$ →



→ (im f aperiodic) → f is FO-definable

→ (im f not aperiodic) → f is not FO-definable → extract all FO-definable information from f

Can we make a quantitative version of Schützenberger-McNaughton-Papert?

Henckell's theorem

Henckell theorem, revisited [Henckell '88]: To each morphism $f: A^* \to M$ we can *effectively* associate a function $g: A^* \to \mathcal{P}(M)$ such that:

- $f(u) \in g(u)$ for all u, and
- g is FO-definable, and
- g is minimal.

Back to the example

$$
\begin{aligned}
f : \{a, b\}^* &\to M \\
u &\mapsto \begin{cases} 1 & \text{if } u = \varepsilon \\ a & \text{if } u \in a(aa)^*, \\ aa & \text{if } u \in (aa)^+, \\ 0 & \text{if } u \text{ contains a `b'} \end{cases}
\end{aligned}
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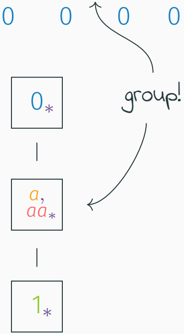

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Goal:

$g : A^* \to \mathcal{P}(M)$ s.t.

- $f(u) \in g(u)$ for all u,
- g is FO-definable, and
- g is minimal.

Back to the example

$$f : \{a, b\}^* \rightarrow M$$
$$u \mapsto \begin{cases} 1 & \text{if } u = \varepsilon \\ a & \text{if } u \in a(aa)^*, \\ aa & \text{if } u \in (aa)^+, \\ 0 & \text{if } u \text{ contains a `b'} \end{cases}$$

$$g : \{a, b\}^* \rightarrow \mathcal{P}(M)$$
$$u \mapsto \begin{cases} \{1\} & \text{if } u = \varepsilon \\ \{a, aa\} & \text{if } u \in a^+, \\ \{0\} & \text{if } u \text{ contains a `b'} \end{cases}$$

Goal:

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Henckell's theorem (bis)

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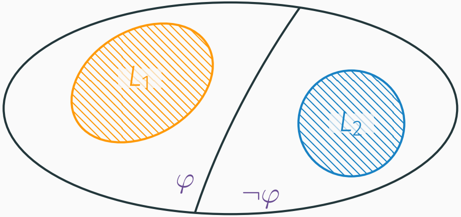
Key technique in the proof: "group saturation".
Observation: If $\operatorname{im} f$ is aperiodic, then the "group saturation" does nothing and $f = g$.

Application: FO-separation



L_1 and L_2 are **FO-separable** whenever there exists $\varphi \in \mathrm{FO}$ such that

$$u \models \varphi \text{ for all } u \in L_1 \qquad v \not\models \varphi \text{ for all } v \in L_2$$

FO-SEPARABILITY:

Input: L_1, L_2 regular languages

Question: Are L_1 and L_2 FO-separable?

Decidable!

- Take $f : A^* \to M$ which recognises L_1 and L_2.
- Build $g : A^* \to \mathcal{P}(M)$.
- Does $g[L_1] \cap g[L_2] = \varnothing$?

Beyond finite words

ω-words: FO cannot capture group-like phenomena
[Perrin '84] (qualitative) & [Place-Zeitoun '16] (quantitative).

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Words indexed by countable ordinals:

Example: bca, $cabc(ab)^\omega$, $(ab^\omega c)^\omega$, etc.

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$$\exists x.\ \mathrm{last}(x) \wedge a(x) \quad \text{where} \quad \mathrm{last}(x) := \forall y.\ y \leqslant x.$$

The word has a last position, and it is an 'a'.

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$a^\omega c b^\omega c a$	$(ab)^\omega b$	a^ω
yes	no	no

FO cannot capture group-like phenomena:
[Bedon '01] (qualitative) & [FoSSaCS 2022] (quantitative).

Example over countable ordinals

	$\cdot$	ε	a	aa	a^ω	$a^\omega a$
empty word →	ε	ε	a	aa	a^ω	$a^\omega a$
finite & odd →	a	a	aa	a	a^ω	$a^\omega a$
finite & even →	aa	aa	a	aa	a^ω	$a^\omega a$
infinite & even suffix →	a^ω	a^ω	$a^\omega a$	a^ω	a^ω	$a^\omega a$
infinite & odd suffix →	$a^\omega a$	$a^\omega a$	a^ω	$a^\omega a$	a^ω	$a^\omega a$

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aa	aa	a	aa	a^ω	$a^\omega a$
a^ω	a^ω	$a^\omega a$	a^ω	a^ω	$a^\omega a$
$a^\omega a$	$a^\omega a$	a^ω	$a^\omega a$	a^ω	$a^\omega a$

empty word → ε

finite & odd → a

finite & even → aa

infinite & even suffix → a^ω

infinite & odd suffix → $a^\omega a$

group! → block $\{a, aa\} \times \{a, aa\}$

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infinite & even suffix	a^ω	a^ω	$a^\omega a$	a^ω	a^ω	$a^\omega a$
infinite & odd suffix	$a^\omega a$	$a^\omega a$	a^ω	$a^\omega a$	a^ω	$a^\omega a$

group!

$$g : u \mapsto \begin{cases} \{\varepsilon\} & \text{if } u \text{ is empty} \\ \{a, aa\} & \text{if } u \text{ is finite} \\ \{a^\omega, a^\omega a\} & \text{if } u \text{ is infinite} \end{cases}$$

Countable words

Words indexed by countable linear orders:

Example: a^ζ, $(ab)^\eta$, etc.

MSO-definable languages $\Leftrightarrow$ languages recognised by some algebras
[Carton-Colcombet-Puppis '18]

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Qualitative results: [Bedon-Rispal '12, Bès-Carton '11 & Colcombet-Sreejith '15].
No quantitative result.

Much harder: FO is not capable to detect:

- groups,
- gaps (ex: a^ω and $a^\omega a^{\omega^*} a^\omega$ are FO-equivalent),
- shuffles.

Conclusion: characterisations of FO

Domain (count. linear order)	Characterisation: non-trivial ... are forbidden	Qualitative	Quantitative
Finite	groups	[Schützenberger '65 & McNaughton-Papert '71]	[Henckell '88]
ω	groups	[Perrin '84]	[Place-Zeitoun '16]
Ordinals	groups	[Bedon '01]	FoSSaCS 2022
Scattered	groups, gaps	[Bès-Carton '11]	ongoing work
Countable	groups, gaps, shuffles	[Colcombet-Sreejith '15]	ongoing work

FoSSaCS 2022: for words over countable ordinals,

- FO-pointlikes are computable
- FO-separability & FO-coverability are decidable