First-order separation over countable ordinals

LX seminar, Bordeaux

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[1] IRIF, CNRS & Univ. Paris
[2] ENS Paris-Saclay → now at LaBRI!

First-order logic (FO)

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$$w \models \exists x. \exists y. x < y \wedge a(x) \wedge b(y)$$

$$\text{iff } w = \cdots \underset{x}{a} \cdots \underset{y}{b} \cdots$$

$$\text{i.e. } w \in A^* a A^* b A^*.$$

FO-definability

FO-DEFINABILITY:

Input: L regular language

Question: Is L definable in FO?

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Fact. For every FO-formula φ, there exists $n \in \mathbb{N}$ such that for all $u \in A^*$, we have:

$$u^n \models \varphi \text{ IFF } u^{n+1} \models \varphi.$$

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- Decidable: [Schützenberger '65 & McNaughton-Papert '71].

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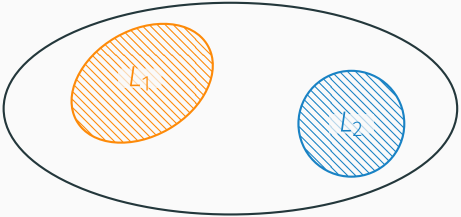

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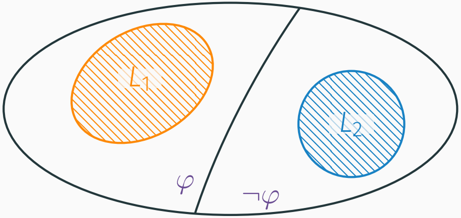

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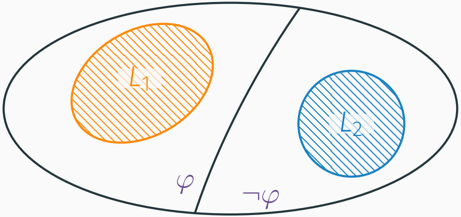
L_1 and L_2 are **FO-separable** whenever there exists $\varphi \in \mathrm{FO}$ s.t. for all $w \in L_1$, $w \vDash \varphi$ and for all $w \in L_2$, $w \nvDash \varphi$.



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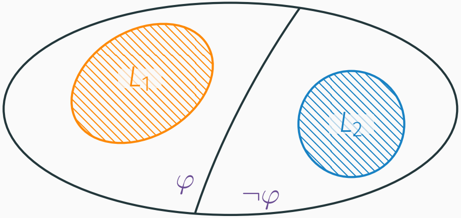
L_1 and L_2 are **FO-separable** whenever there exists $\varphi \in$ FO s.t. for all $w \in L_1$, $w \models \varphi$ and for all $w \in L_2$, $w \not\models \varphi$.

Example: $b^+(aa)^+$ and $(aa)^+$ can be separated by $\exists x.\, b(x)$.



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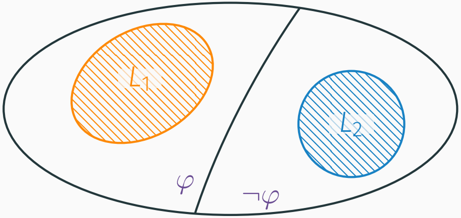

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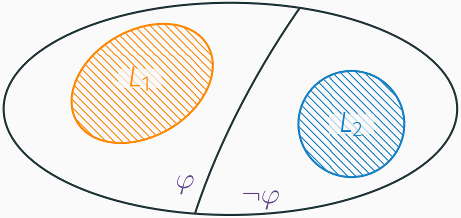

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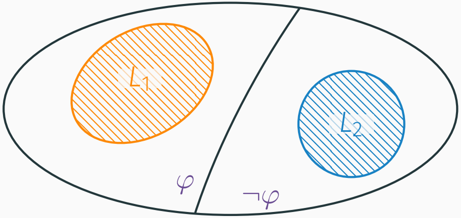

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reduces to
$(L \mapsto (L, A^* \setminus L))$

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FO-separability is **decidable**: [Henckell '88 & Almeida '96], and [Place-Zeitoun '16].

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Domain (countable linear order)	FO-definability	FO-separability
ω	dec. [Perrin '84]	dec. [Place-Zeitoun '16]

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Ordinals	dec. [Bedon '01]	dec. [this talk!]
Scattered	dec. [Bès-Carton '11]	? [future work]
Countable	dec. [Colcombet-Sreejith '15]	

FO-definability: Schützenberger-McNaughton-Papert

$$
\begin{array}{ccc}
\mathrm{Reg} = \mathrm{MSO} & \overset{\text{recognises}}{\longleftrightarrow} & \text{fin. monoids} \\
\uparrow & & \uparrow \\
\mathrm{FO} & \longleftrightarrow & \text{aperiodic fin. monoids}
\end{array}
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Schützenberger-McNaughton-Papert theorem

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Schützenberger-McNaughton-Papert theorem

M is **aperiodic** when every group $G \subseteq M$ is trivial.

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$$
\begin{tikzcd}
 & & & & s^{n+1} \arrow[dr, bend left] & \\
1 \arrow[r] & s \arrow[r] & \cdots \arrow[r] & s^n \arrow[ur, bend left] & & s^{n+2} \arrow[dl, bend left] \\
 & & & & s^{n+3} \arrow[ul, bend left] &
\end{tikzcd}
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Not aperiodic!

FO-definability: Schützenberger-McNaughton-Papert

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$$1 \longrightarrow s \longrightarrow \cdots \longrightarrow s^n \circlearrowleft$$

Aperiodic!

FO-definability: words of even length

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$$\begin{array}{rccc} \varphi: & a^* & \to & \mathbb{Z}/2\mathbb{Z} \\ & w & \mapsto & |w| \mod 2 \end{array}$$

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$$(aa)^* = \varphi^{-1}[\bar{0}]$$

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Every monoid recognising $(aa)^*$ must contain a non-trivial group
$\rightsquigarrow$ not FO-definable.

FO-separability: example

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\begin{aligned}
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L_1 and L_2 are FO-separated by $\exists x.\, b(x)$.

L_2 and L_3 are not FO-separable (Schützenberger-McNaughton-Papert thm).

FO-separation: Henckell & Almeida

L_1, L_2 recognised by $\varphi\colon A^* \to M$.

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Theorem [Henckell '88 & Almeida '96]: There exists a computable object $\mathrm{Sat}^+(A) \subseteq \mathcal{P}(M)$ such that:

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for every $m_1 \in \varphi[L_1]$ and $m_2 \in \varphi[L_2]$, we have $\{m_1, m_2\} \notin \downarrow \mathrm{Sat}^+(A)$.

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Corollary: FO-separability is decidable.

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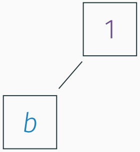

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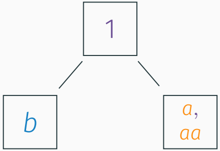

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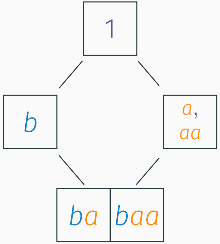

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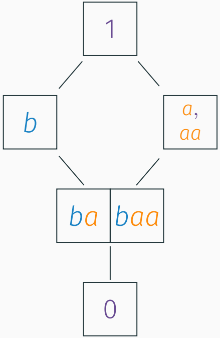

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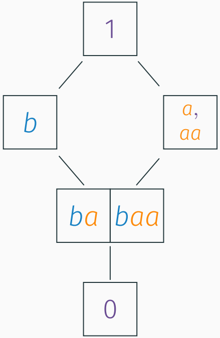

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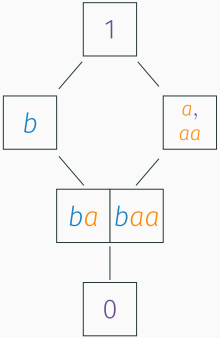

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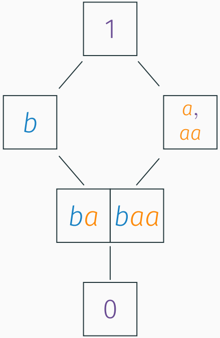

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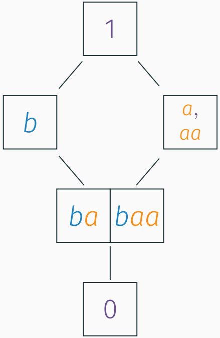

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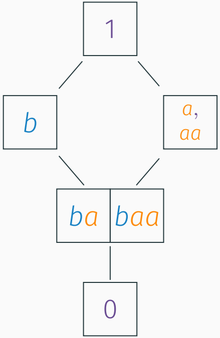

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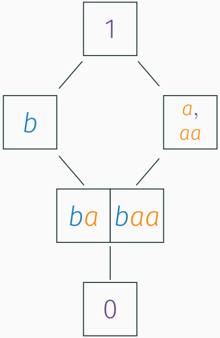

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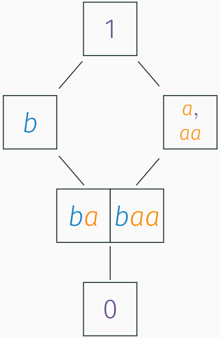

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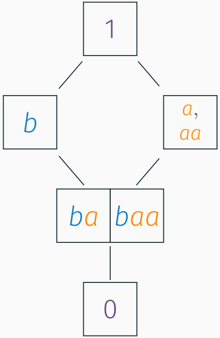

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Summary: finite words

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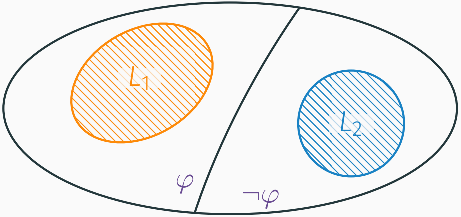
Question: Are L_1 and L_2 FO-separable?



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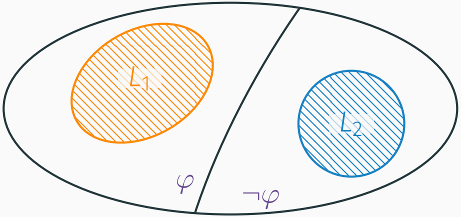

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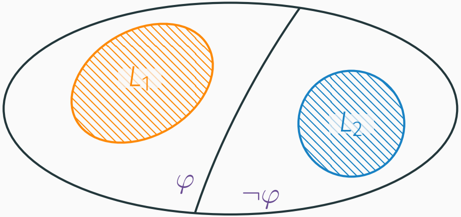

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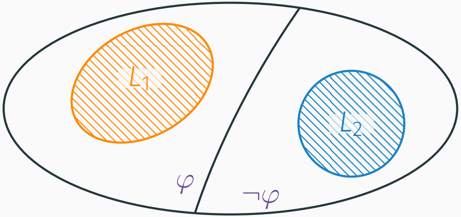

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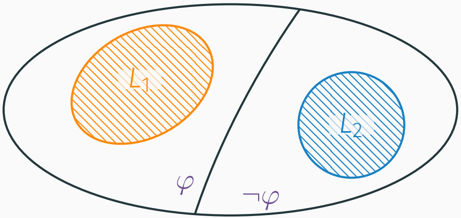

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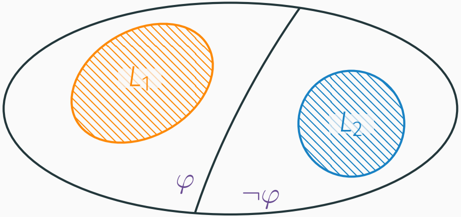

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Transfinite word: word indexed by a countable ordinal. Notation: A^{ord}.

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Example: A^{ord} all transfinite words over A...

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Theorem [Bedon '01]: $L \subseteq A^{\mathrm{ord}}$ is FO-definable iff it is recognised by a finite aperiodic ordinal monoid.

FO-definability: co-example

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To decide if a word $u \in a^{\text{ord}}$ is in L_{even},
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$\rightsquigarrow$ L_{even} not FO-definable.

FO-separability: generalising Henckell's theorem

Finite words:

To decide FO-separability, compute the "saturation".

Saturation: algebraic structure, closed under:

- product,
- union of groups.

Notation: $\mathrm{Sat}^{+}(A)$

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To go from finite words to transfinite words, just add an ω everywhere.

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$\text{Sat}^{\text{ord}+}(A)$ contains: $\{a\}$, $\{aa\} = \{a\} \cdot \{a\}$, $\{a^\omega\} = \{a\}^\omega$, $\{a, aa\} =$ group!,
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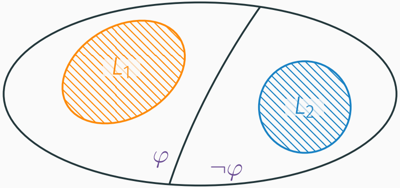

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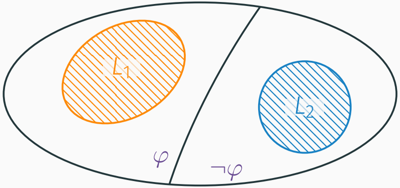

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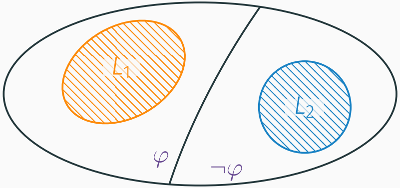

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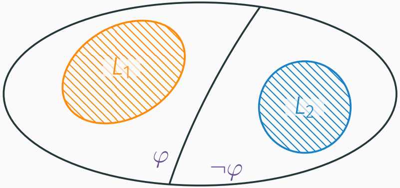

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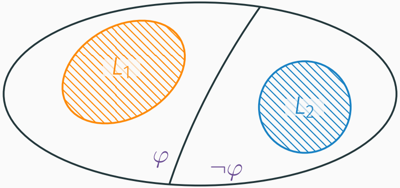

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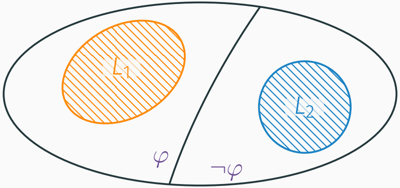

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"Groups are monoids in which left and right multiplication is bijective."

Glimpse of the proof for finite words (2)

Fact. For all A-generated monoid M:

i. $\exists a \in A,\ a \cdot M \subsetneq M$, or
ii. $\exists a \in A,\ M \cdot a \subsetneq M$, or
iii. M is a group.

Lemma. For every alphabet A, either:

i. $\exists a \in A,\ a \cdot \mathrm{Sat}^+(A) \subsetneq \mathrm{Sat}^+(A)$, or
ii. $\exists a \in A,\ \mathrm{Sat}^+(A) \cdot a \subsetneq \mathrm{Sat}^+(A)$, or
iii. $\mathrm{Sat}^+(A)$ has some maximum.

monoids $\rightsquigarrow$ "monoids with union of groups"

"Groups are monoids in which left and right multiplication is bijective."

Groups and monoids are symmetric objects.

Towards a proof for transfinite words...

Proof for transfinite words. We want to study "group-like phenomenon" in ordinal monoids, which are very not symmetric.

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Proof for transfinite words. We want to study "group-like phenomenon" in ordinal monoids, which are very not symmetric.

$\rightsquigarrow$ We need to think about groups "asymmetrically". It's not easy!

Lemma. For every alphabet A, either:

we gain some information when...

i. $\exists a \in A,\ a \cdot \mathrm{Sat}^{\mathrm{ord}+}(A) \subsetneq \mathrm{Sat}^{\mathrm{ord}+}(A)$, or $\leftsquigarrow$ reading an 'a'

ii. $\mathrm{Sat}^{\mathrm{ord}+}(\mathrm{Sat}^{+}(A)^{\omega}) \subsetneq \mathrm{Sat}^{\mathrm{ord}+}(A)$, or $\leftsquigarrow$ reading any ω-word!

iii. $\mathrm{Sat}^{\mathrm{ord}+}(A)$ is an $\mathcal{L}$-trivial $\mathcal{R}$-class. $\leftsquigarrow$ easy base case

Conclusion

Lemma. For every alphabet A, either:

i. $\exists a \in A,\ a \cdot \mathrm{Sat}^{\mathrm{ord}+}(A) \subsetneq \mathrm{Sat}^{\mathrm{ord}+}(A)$, or
ii. $\mathrm{Sat}^{\mathrm{ord}+}(\mathrm{Sat}^{+}(A)^{\omega}) \subsetneq \mathrm{Sat}^{\mathrm{ord}+}(A)$, or
iii. $\mathrm{Sat}^{\mathrm{ord}+}(A)$ is an $\mathcal{L}$-trivial $\mathcal{R}$-class.

we gain some information when...
⇜ reading an 'a'
⇜ reading any ω-word!
⇜ easy base case

Domain (countable linear order)	FO-DEFINABILITY	FO-SEPARABILITY
Finite	dec. [Schützenberger '65 & McNaughton-Papert '71]	dec. [Henckell '88 & Almeida '96]
ω	dec. [Perrin '84]	dec. [Place-Zeitoun '16]
Ordinals	dec. [Bedon '01]	dec. ← new result!
Scattered	dec. [Bès-Carton '11]	??
Countable	dec. [Colcombet-Sreejith '15]	??