Solving parity games

Universal trees and hierarchical decompositions

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Parity games

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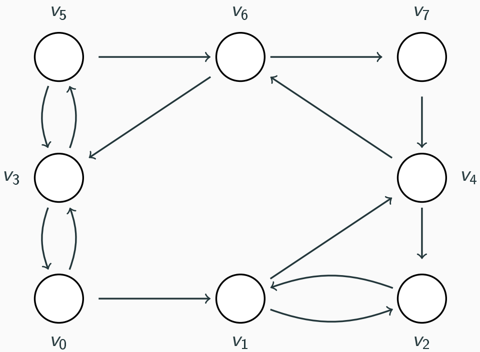
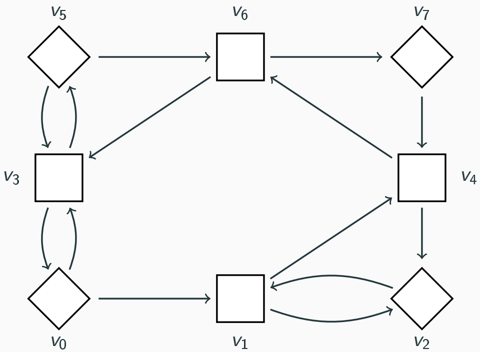


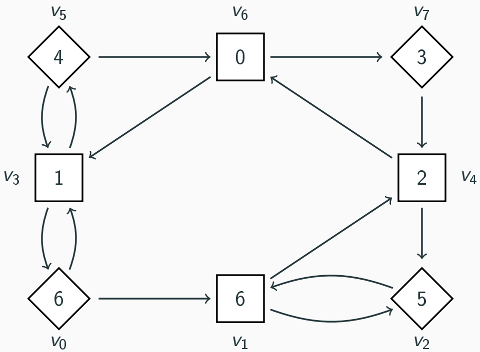
$$\mathcal{G} = \langle V, E \rangle$$

Parity games



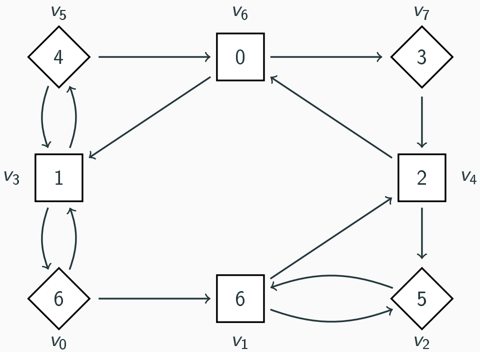
$$\mathcal{G} = \langle V, E, V_{\mathrm{Even}}, V_{\mathrm{Odd}} \rangle$$

Parity games



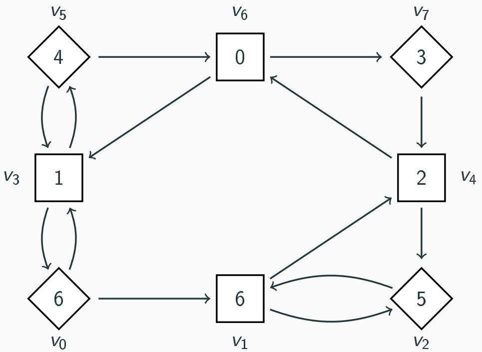
$$\mathcal{G} = \langle V, E, V_{\mathrm{Even}}, V_{\mathrm{Odd}}, \ \pi : V \to \mathbb{N} \rangle$$
$$\pi(V) \subseteq [\![0, d]\!]$$

Plays



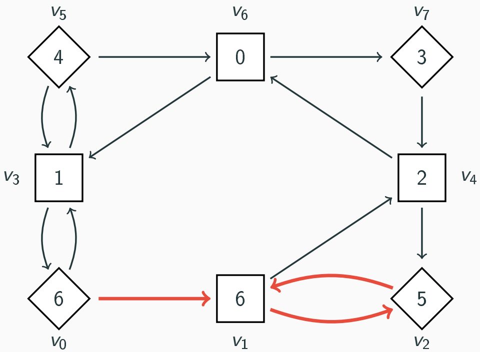
Play: $(v_i)_{i \in \mathbb{N}}$ s.t. $\forall i,\ (v_i, v_{i+1}) \in E$

Plays



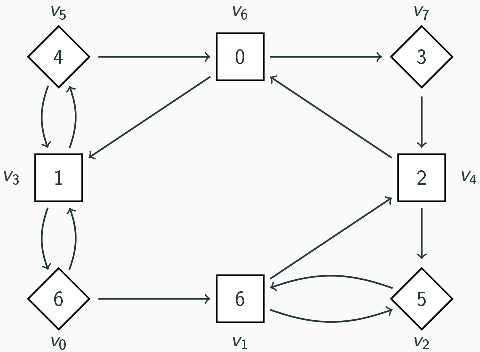
$(v_i)_{i\in\mathbb{N}}$ **winning for Even** iff $\limsup_{i\in\mathbb{N}} \pi(v_i)$ is even.

Plays



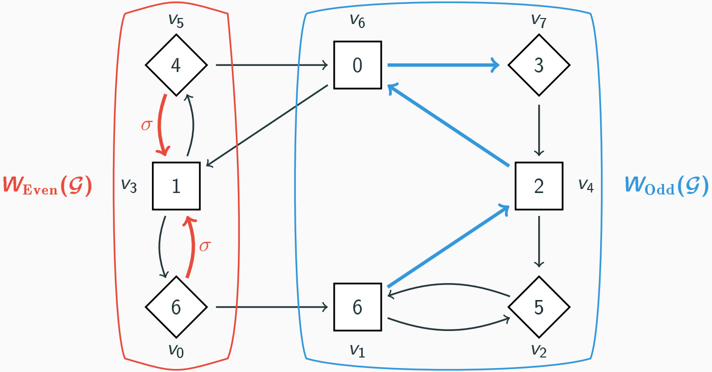
Eg: $v_0 \cdot (v_1 \cdot v_2)^\omega$ is winning for Even.

Strategies



(Memoryless) **strategy for Even:** $\sigma : u \mapsto v$
where $u \in V_{\mathrm{Even}}$ and $(u, v) \in E$.

Strategies



Memoryless Determinacy Theorem:

From every vertex, one of the two players has a memoryless strategy such that every play consistant with it is winning for her.

Decision problem

Solving parity games:

Data: $(\mathcal{G}, v)$

Question: $v \in W_{\mathrm{Even}}(\mathcal{G})$?

- decidable
- NP∩co-NP
- UP∩co-UP [Jurdziński, 1998]
- maybe in P: **open question!**

Decision problem

Solving parity games:

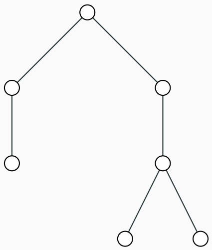
Data: $(\mathcal{G}, v)$

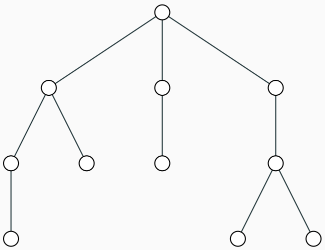
Question: $v \in W_{\mathrm{Even}}(\mathcal{G})$?

- $O(n^{d+O(1)})$ [Zielonka, 1998]
- $O(n^{O(\sqrt{d})})$ [Björklund-Sandberg-Vorobyov, 2003]
- $O(n^{O(\lg d)})$ [Calude-Jain-Khoussainov-Li-Stephan, 2017]

Trees

Embedding





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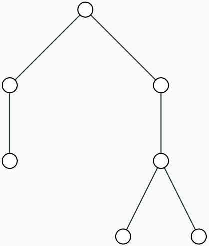

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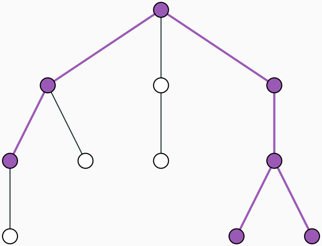

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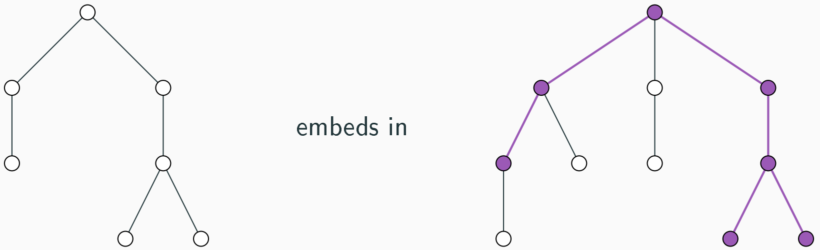


(n, d)-**universal tree:** [Fijalkow, 2018]
Every tree with $\leq n$ leaves and of height $\leq d$ embeds in it.
[Czerwiński-Daviaud-Fijalkow-Jurdziński-Lazić-Parys, 2019]

Recursive algorithms: tree of recursive calls

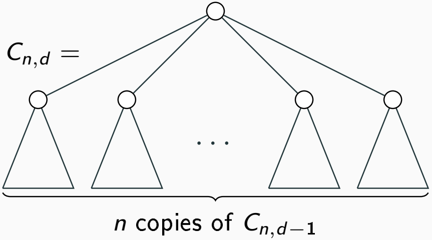
Zielonka '98

Parys '18

LSW '19

Recursive algorithms: tree of recursive calls

Zielonka's algorithm (1998)



Parys '18

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Recursive algorithms: tree of recursive calls

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Parys' algorithm (2018)

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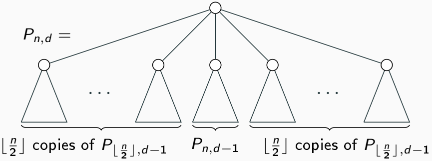

Recursive algorithms: tree of recursive calls

Zielonka '98

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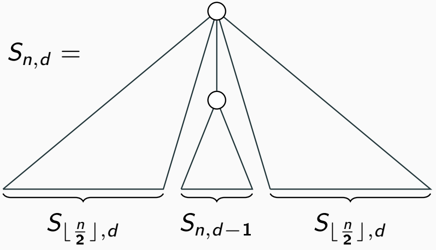
Lehtinen-Schewe-Wojtczak's algo. (2019)



[Jurdziński-Lazić, 2017]

Recursive algorithms: tree of recursive calls

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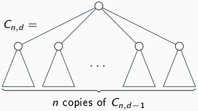
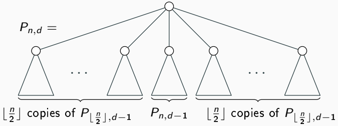
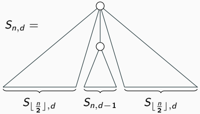


Parys '18



LSW '19



Those trees are (n, d)-universal!

Generalization?

Let $\texttt{Solve}(\mathcal{G}, d, \mathcal{T})$ be s.t. $\mathcal{T}$ is the tree of recursive calls and:

- $\mathcal{T} = C_{n,d} \rightsquigarrow$ Zielonka
- $\mathcal{T} = P_{n,d} \rightsquigarrow$ Parys
- $\mathcal{T} = S_{n,d} \rightsquigarrow$ LSW

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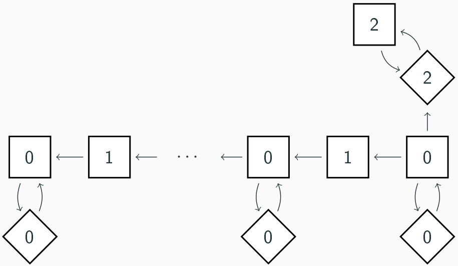
Question: If $\mathcal{T}$ is (n, d)-universal, then $\texttt{Solve}(\mathcal{G}, d, \mathcal{T}) = W_{\mathrm{Even}}(\mathcal{G})$?

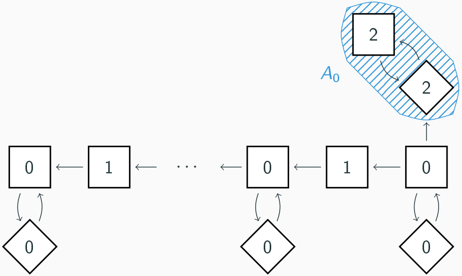
Hierarchical decompositions

Hierarchical decomposition



k-legged wild goat parity game and its hierarchical decomposition
[Daviaud-Jurdziński-Lehtinen, 2018]

Hierarchical decomposition



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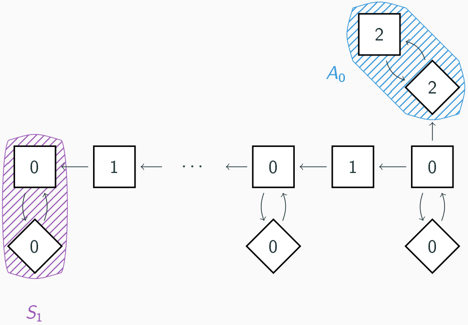

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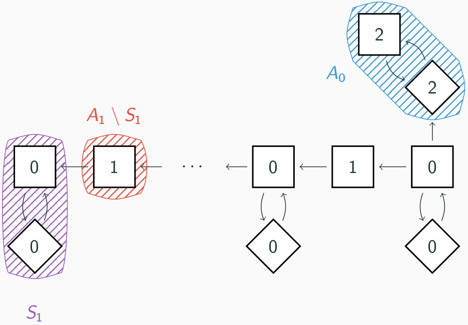

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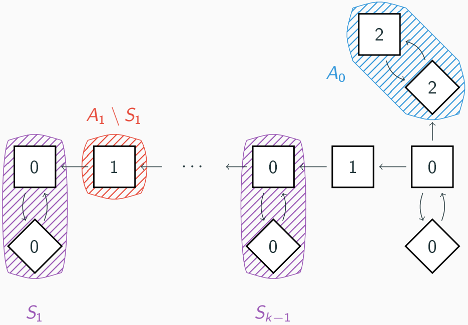

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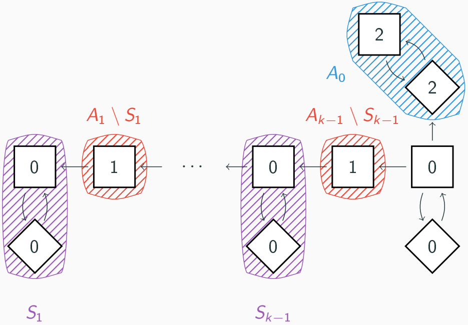

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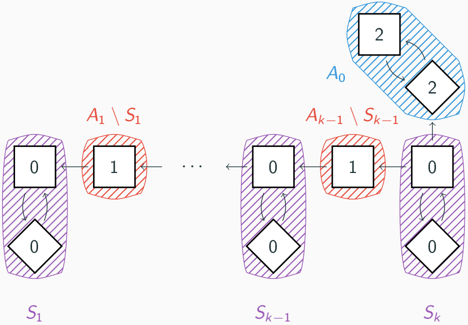

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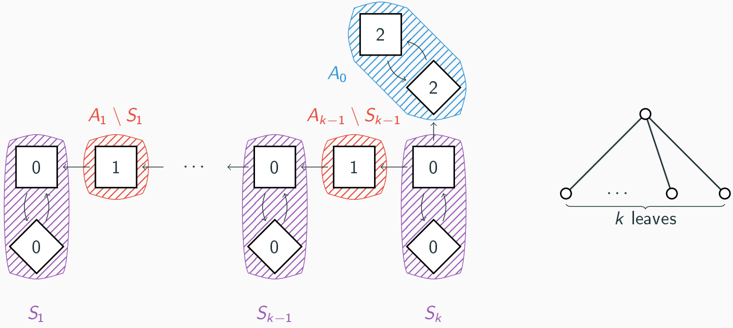
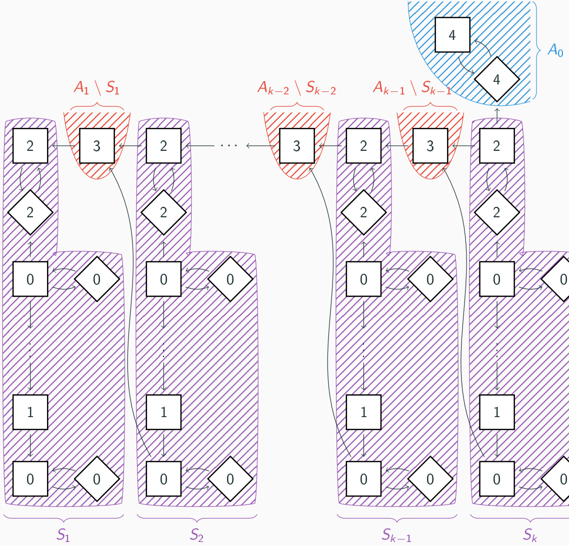
Hierarchical decomposition tree



k-legged wild goat parity game and its hierarchical decomposition tree [Daviaud-Jurdziński-Lehtinen, 2018]

Another example



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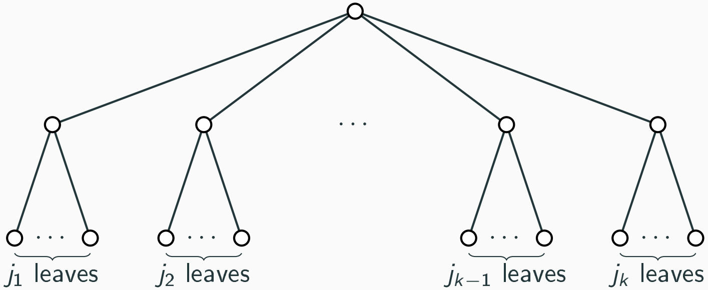

It was easy!

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The HDT $\mathcal{T}_\mathcal{G}$ of a game represents its complexity.
If $\mathcal{T}_\mathcal{G} \hookrightarrow \mathcal{T}$, then $\texttt{Solve}(\mathcal{G}, d, \mathcal{T}) = W_{\mathrm{Even}}(\mathcal{G})$.

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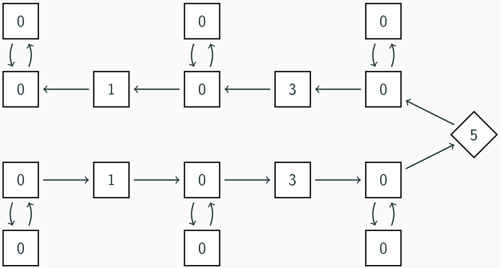
If $\mathcal{T}_{\mathcal{G}} \hookrightarrow \mathcal{T}$, then $\texttt{Solve}(\mathcal{G}, d, \mathcal{T}) = W_{\mathrm{Even}}(\mathcal{G})$.

We "just" need to check that if $\mathcal{G}'$ is a subgame of $\mathcal{G}$, then $\mathcal{T}_{\mathcal{G}'} \hookrightarrow \mathcal{T}_{\mathcal{G}}$.

Wait...



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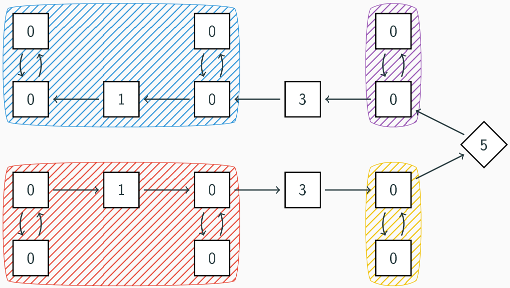

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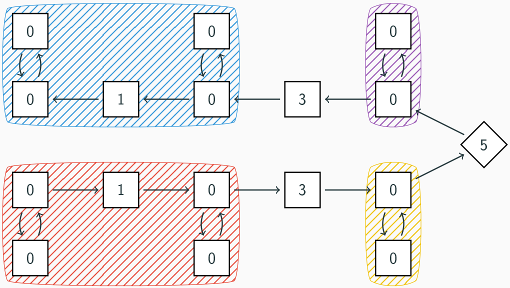

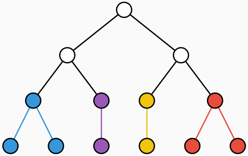

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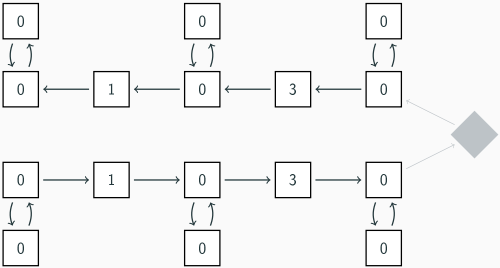

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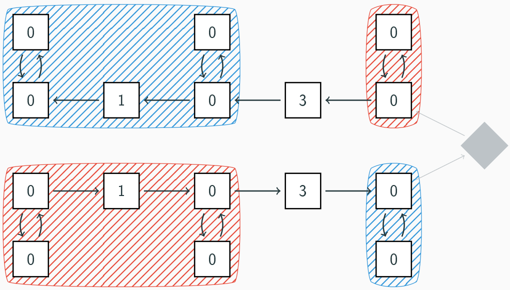

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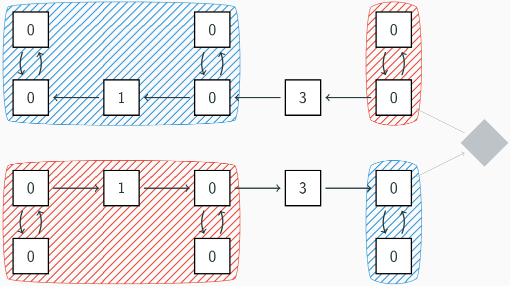

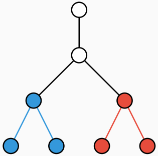

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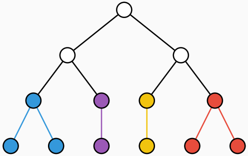

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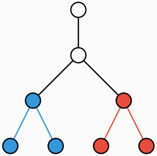

Wait...

Solution

Change the definition of the HDT
$\rightsquigarrow \mathcal{G}$ has **a set of** HDT.

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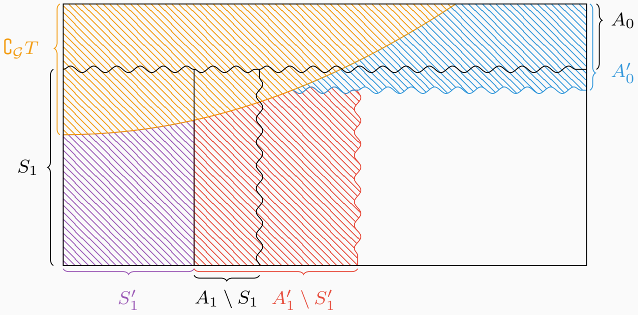

Conclusion

For all games $\mathcal{G}$, there exists $\mathcal{T}_{\mathcal{G}}$ such that for all $\mathcal{T}$, if $\mathcal{T}_{\mathcal{G}} \hookrightarrow \mathcal{T}$ then $\texttt{Solve}(\mathcal{G}, d, \mathcal{T}) = W_{\mathrm{Even}}(\mathcal{G})$.

Youpi!

Questions?