Definability & separability of regular languages in first-order logic

MPRI internship defense

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September 3, 2021

First-order logic (FO)

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i.e. $w \in A^* a A^* b A^*$.

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Input: L regular language

Question: Is L definable by a first-order formula?

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- Non-trivial: $(aa)^*$ is not FO-definable.
- Decidable: [Schützenberger '65 & McNaughton-Papert '71].

FO-separability

L_1 and L_2 are FO-**separable** whenever there exists $\varphi \in \mathrm{FO}$ s.t. for all $w \in L_1$, $w \models \varphi$ and for all $w \in L_2$, $w \not\models \varphi$.

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reduces to

$(L \mapsto (L, A^* \setminus L))$

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FO-separability (bis)

FO-separability is **decidable**:

[Henckell '88 & Almeida '96], and [Place-Zeitoun '16].

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Ordinals	dec. [Bedon '01]	dec. [my internship!]
Scattered	dec. [Bès-Carton '11]	? [future work]
Countable	dec. [Colcombet-Sreejith '15]	

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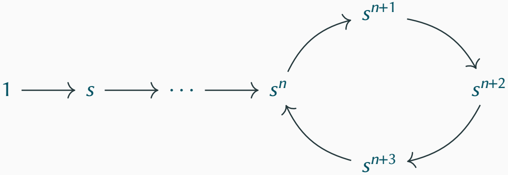

Not aperiodic!

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Every monoid recognising $(aa)^*$ must contain a non-trivial group
⇝ not FO-definable.

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L_2 and L_3 are not FO-separable (Schützenberger-McNaughton-Papert thm).

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Theorem [Henckell '88 & Almeida '96]: There exists a computable submonoid $\mathrm{Sat}(M) \subseteq \mathscr{P}(M)$ such that:

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Corollary: FO-separability is decidable.

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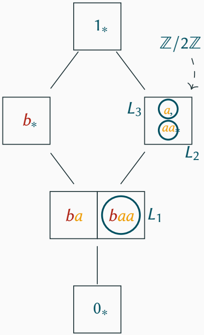

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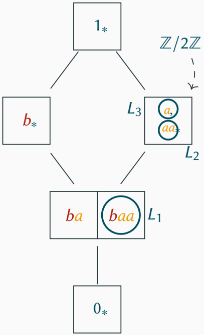

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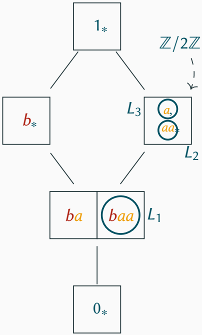

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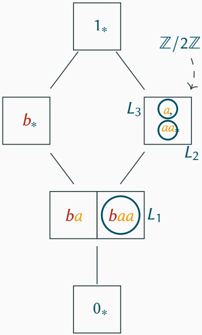

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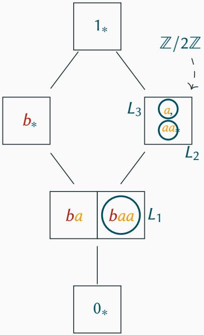

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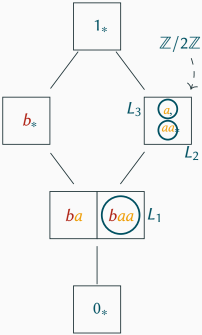

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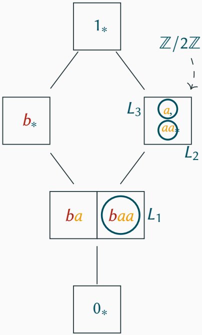

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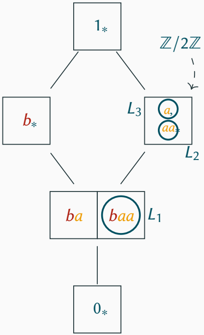

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Saturation: definition & example

Proof (correctness):
If $X \in \mathrm{Sat}(M)$, then the elements of X cannot be distinguished by FO. (Easy!)

Henckell's theorem: "$\mathrm{Sat}(M)$ is the collection of subsets of M whose points cannot be distinguished by FO (pointlikes)".

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Proof (completeness):
If the elements of X cannot be distinguished by FO, then $X \in \mathrm{Sat}(M)$. (Not easy!)

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Proof sketch: completeness *(new proof!)*

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- $\varphi(a) \cdot \operatorname{Sat}(M) \subsetneq \operatorname{Sat}(M)$ for some $a \in A$, or
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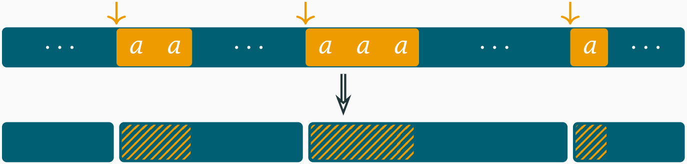

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- Example of first-order formula: $\neg\exists x.\, \mathrm{last}(x)$.

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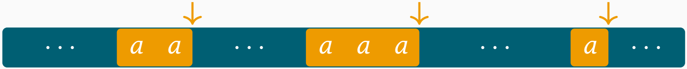

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For ordinals, knowing that $\mathrm{Sat}^{\mathrm{ord}}(S) \cdot \varphi(a) \subsetneq \mathrm{Sat}^{\mathrm{ord}}(S)$ for some $a \in A$ is useless.

Magnificent solution!

Lemma: Either

i. $\varphi(a) \cdot \mathrm{Sat}^{\mathrm{ord}}(S) \subsetneq \mathrm{Sat}^{\mathrm{ord}}(S)$ for some $a \in A$, or

ii. $\mathrm{Sat}^{\mathrm{ord}}(\mathrm{Sat}^{\omega}(S)) \subsetneq \mathrm{Sat}^{\mathrm{ord}}(S)$, or

iii. $\mathrm{Sat}^{\mathrm{ord}}(S)$ is a $\mathscr{L}$-trivial $\mathscr{R}$-class.

Conclusion

Domain (countable linear order)	FO-definability	FO-separability
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ω	dec. [Perrin '84]	dec. [Place-Zeitoun '16]
Ordinals	dec. [Bedon '01]	**dec.** ← ***new!***
Scattered	dec. [Bès-Carton '11]	?? ← ***future work***
Countable	dec. [Colcombet-Sreejith '15]	

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